On Some Aspects of Summability and its Application

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BY

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THESIS SECTION

CERTIFICATE

This is to certify that the contents of this thesis entitled, " ON SOME ASPECTS OF SUMMABILITY AND ITS APPLICATION ", is an original research work of Mr. Satish Kumar, done under my supervision.

I further certify that the work of this thesis, either partly or fully has not been submitted to any other institution for the award of any other degree.

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to the loving memory of my

FATHER

ON SOME ASPECTS OF SUMMABILITY AND ITS APPLICATION

P R E F A C E

The present Thesis entitled, " On Some Aspects of Summability and its Application ", is the outcome of my researches that I have been pursuing for the last three years, under the esteemed supervision of Dr. Z.U.Ahmad, M.Sc.,D.Phil.,D.Sc., Reader, Department of Mathematics, Aligarh Muslim University, Aligarh.

It has been my proud privilege to have accomplish my researches under the able supervision of Dr. Z. U. Ahmad who has made valuable contributions in the Theory of Absolute Summability. I have great pleasure in taking this opportunity of acknowledging my deep sense of gratitude and high indebtedness to Dr. Ahmad for his inspiring guidance, constant help and encouragement throughout to complete this thesis.

The Thesis consists of eight chapters including Chapter Zero which concerns with note on some conventions used in the body of the thesis. In Chapter I, besides some further notations and definitions, we give brief résumé of more important results which have interconnections with our investigations. Chapter II contains a theorem on $|N,p_n|$-summability factors of infinite series and three theorems on power series and Fourier series, which are obtained as a consequence of the applications of the first theorem. Chapter III deals with the study of $|N,p_n,q_n| \Longrightarrow$ absolute convergence factors, while Chapter IV concerns with certain results on $|N,p_n,q_n|$-summability factors of infinite series. Chapter V is devoted to the study of absolute matrix summability factors of a Fourier series. Chapter VI contains certain results on Tauberian theorems for $|J,p_n|_k$-summability. The VIIth and the last chapter deals with the study of (J,p_n)-summability of the derived series of a Fourier series. Towards the end we give a comprehensive bibliography of various publications which have been referred to in the present thesis.

It may be mentioned here that some portion of the thesis, in the form of research papers, has been communicated for publication in various international mathematical journals.

I take this opportunity to extend my sincere thanks to

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SKGupta

Dated : 8-12-1981 (Satish Kumar)

CONTENTS

Chapter Zero

NOTE ON CONVENTIONS

Here we state a few conventions which are not emphasised in the following chapters.

0.1 SUMMATION CONVENTIONS :

Σ, written without limits, usually denotes $\sum\limits_{0}^{\infty}$, or $\sum\limits_{1}^{\infty}$ if a term of zero rank is not defined.

$\sum\limits_{n} a_n$ is the sum of all a_n's which are defined.

0.2 BINOMIAL COEFFICIENTS :

For $n = 0,1,2,\ldots$, A_n^{α} is defined by the identity :

$$\sum_{n=0}^{\infty} A_n^{\alpha} x^n = (1-x)^{-\alpha-1} \qquad (|x| < 1),$$

$$A_n^{\alpha} = \frac{(\alpha+1)\ (\alpha+2)\ldots(\alpha+n)}{\underline{|n}} \qquad (\alpha > -1),$$

$$A_n^{-\alpha} = 0. \qquad (n \geq \alpha,\ \alpha = 1,2,\ldots).$$

0.3 CONSTANT :

K denotes, throughout, an absolute constant independent of the variable under consideration, but is not necessarily the same at each occurrence.

K is also used as a symbol for the work 'Conservative'.

0.4 SYMBOLS T, AK, AT :

T, AK and AT are used to symbolise the word 'regular' and the phrases 'Absolutely Conservative' and 'Absolutely regular' respectively.

0.5 ORDER NOTATION O and o :

If g is a positive function of a variable which tends to a given limit, we shall write

$$f = o(g), \quad \text{if} \quad f/_{g} \to o \ ,$$

and

$$f = O(g), \quad \text{if} \quad |f| < Kg.$$

In particular

$$f = o(1) \text{ means that } f \to o,$$

and

$$f = O(1) \text{ means that } f \text{ is bounded.}$$

0.6 SYMBOLS $\subseteq$, $\subset$ and $\sim$:

Given two methods of summability (or absolute summability), P and Q, we write $P \subseteq Q$, or $Q \supseteq P$ for 'P includes Q' or 'Q is included in P' to mean that every sequence summable by P is also summable by Q.

If $P \subseteq Q$ and $Q \subseteq P$, the two methods P and Q are said to be equivalent and we write $P \sim Q$.

If $P \subseteq Q$ and there exists a sequence which is summable Q but not summable P, then we write $P \subset Q$.

0.7 FINITE DIFFERENCES :

For any sequence $\{f_n\}$, we write

$$\Delta f_n = f_n - f_{n+1}, \quad \Delta^0 f_n = f_n$$

$$\Delta^2 f_n = \Delta(\Delta f_n) = \Delta(f_n - f_{n+1})$$

$$\Delta^k f_n = \Delta(\Delta^{k-1} f_n) \quad (k = 1, 2, \ldots)$$

and

0.8 CONVEX AND QUASI-CONVEX SEQUENCES :

The sequence $\{f_n\}$ is said to be convex if $\Delta^2 f_n \geq 0$, $n = 1,2,\ldots$ ([108], p.58). It is well known that if $\{f_n\}$ is bounded and convex, then

$$f_n \downarrow \ , \ n \Delta f_n \to 0, \ n \to \infty, \text{ and}$$

$$\sum_{n=1}^{\infty} (n+1) \ \Delta^2 f_n < \infty .$$

A sequence $\{f_n\}$ is said to be quasi-convex ([108], p.58), if

$$\sum_{n=1}^{\infty} (n+1) \ | \ \Delta^2 f_n \ | < \infty .$$

It is clear from the above result that every bounded convex sequence is quasi-convex. However, the converse need not be true. Contrary to what we have for convex sequences a null quasi-convex sequence $\{f_n\}$ need not be monotonic decreasing. It is, however, of bounded variation and it satisfies the condition,

$$n \ \Delta f_n \to 0, \ n \to \infty .$$

0.9 BOUNDED VARIATION :

By $\{t_n\} \in BV$, we mean that the sequence $\{t_n\}$ is of bounded variation, that is to say

$$\Sigma \mid t_n - t_{n-1} \mid \leq K ,$$

which is the same as

$$\Sigma \mid \bar{\Delta} t_n \mid \leq K .$$

By ' $t_n \in BV^k$ ', $k \geq 1$, we mean that

$$\Sigma\, n^{k-1} \mid \bar{\Delta} t_n \mid^k \leq K.$$

By '$f(x) \in BV(o,1)$' we mean that $f(x)$ is a function of bounded variation in the interval $(o,1)$, that is

$$\int_o^1 \mid \frac{d}{dx} \{f(x)\} \mid dx \leq K ,$$

and by '$f(x) \in BVk\ (o,1)$', for $k \geq 1$, $o < c < 1$, we mean that

$$\int_o^1 (1-x)^{k-1} \mid \frac{d}{dx} \{f(x)\} \mid^k dx \leq K ,$$

so that '$f(x) \in BV^1(o,1)$' is the same as '$f(x) \in BV(o,1)$'.

0.10 INTEGRAL PART OF x :

$[x]$ denotes the algebraically greatest integer not exceeding x.

Apart from these, all notations and conventions of Chapter zero? I will be adhered to throughout the rest of the Thesis without specific mention, unless otherwise stated.

Chapter I

INTRODUCTION

1.1. With the appearance of CAUCHY's Analyse Algebrique in 1821, (See [19]), and ABEL's researches on binomial series in 1826, (See [1]), the old hazy notion of convergence of infinite series was put on sound foundation. The concept of convergence as given by Cauchy devided the infinite series into two classes, viz., those which have a finite (and unique) sum in this sense and those that fail to have. But there remained to be precisely apprehended the distinction between properly divergent series and series with finitely oscillatory partial sums. Towards the end of the last century the classical concept of convergence was generalised so as to bring in the purview of a sound mathematical interpretation a large variety of divergent series of the second kind. This generalised concept which is known as Summability* is a generalisation of the notion

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* For information about Summability and its applications, reference may be made, e.g., to FORD [39], HARDY [40], KNOPP [49], PATI [76], PETERSEN [80], PEYERIMHOFF [85], POWELL and SHAH [85], SINHA [94], Szász [98], ZELLER and BEEKMANN [107], ZYGMUND [108].

of Cauchy convergence in the sense that the partial sum is to be replaced by a suitable transform of it in a certain prescribed manner. For the pioneering researches that led to the " Theory of Summability ", the credit goes inter-alia to Hölder, Cesàro, Riemann, Hausdorff, Borel and others, (Cf. HARDY [40]).

Just as the concept of convergence gave rise to that of summability, so also, in more recent times, the notion of absolute convergence led to the formulation of, what is called, the " Theory of Absolute Summability ".†

The present Thesis covers authors investigations concerning 'Some aspects of summability and its application'. Before presenting a résumé of the earlier researches in the light of which various new results have been obtained by the author, it seems desirable to give here certain fundamental concepts, definitions and notations which will be required in the sequel.

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† For information about Absolute Summability and its applications, reference may be made, e.g., to PATI [76], SINHA [94], BEEKMANN and ZELLER [107].

1.2. Some of the most familiar methods of summability and absolute summability with which we shall be concerned in the sequel are those of Abel, Cesàro, Nörlund etc. It may, however, be mentioned that all these methods can be derived from the two basic general methods (cf. [40], p. 42), viz.,

(i) the T-methods ,

(ii) the ϕ-methods.

A T-method is based upon the formation of a sequence of auxiliary means, $\{t_n\}$, defined by the sequence-to-sequence transformation :

$$(1.2.1) \qquad t_n = \sum_{k=0}^{n} a_{n,k}\, s_k \qquad (n = 0,1,2,\ldots),$$

a_{nk} being the elements of the n^{th} row and k^{th} column of the Toeplitz matrix $T = (a_{nk})$, and s_k, the k^{th} partial sum of a given infinite series $\Sigma\, a_n$.

Other types of transformations under this category are the sequence-to-series transformation, the series-to-series transformation, and the series-to-sequence transformation, with which we are not concerned here.

A ϕ-method is based upon the formation of the functional transformation, $t(x)$, defined by sequence-to-function transformation :

$$(1.2.2) \quad t(x) = \Sigma\, \phi_n(x)\, s_n ,$$

or, by function-to-function transformation :

$$(1.2.3) \quad t(x) = \int_0^\infty \phi(x,y)\, s(y)\, dy,$$

where x is a continuous parameter and the function $\phi_n(x)$ (or $\phi(x,y)$) is defined over a suitable interval of x (or of x and y). Similarly, we have one more transformation under this category, with which we are not concerned.

The series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be summable to a finite number s, by a T-method, or a ϕ-method, according as the sequence $\{t_n\}$, or the function $t(x)$, tends to s, as n tends to infinity, or as x tends to an appropriate limit, depending upon the method, that is,

$$\lim_{n \to \infty} t_n = s, \quad \text{or} \quad \lim_{x \to x_0} t(x) = s.$$

The series Σa_n, or the sequence $\{s_n\}$, is said to be absolutely summable by a T-method, or simply summable $|T|$, if $\{t_n\} \in BV$, and it is said to be summable $|T|_k$, $k \geq 1$, if $\{t_n\} \in BV^k$. $|T|_1$ is the same as $|T|$.

Absolute summability by a Φ-process, or summability $|\Phi|$, is similarly defined, with the obvious difference that, in this case, $t(x) \in BV(A, \rho)$, where (A, ρ) is a suitable interval of variation of the continuous variable x; and for the summability $|\Phi|_k$, $k \geq 1$, we have $t(x) \in BV^k(A, \rho)$. $|\Phi|_1$ is the same as $|\Phi|$.

1.3. A method of summability P is K(or AK), if the convergence (or bounded variation) of the sequence $\{s_n\}$ (of the partial sums of the given series Σa_n) implies the summability P (or $|P|$) of Σa_n in each case, and P is T (or AT), if P is K(or AK) and also preserves sums of convergent series. It has been observed by MORLEY [65] that a method may be AK without being K.

Necessary and sufficient conditions that a matrix method be AK were first obtained by MEARS [59] in 1937, and functional Analytic proofs of equivalent results were given later on by KNOPP and LORENTZ [50] and SUNOUCHI [96].§

The necessary and sufficient conditions for the transformation (1.2.1) to be K, are (Cf. [40]) :

$$(1.3.1)\qquad \begin{cases} \text{(i)} & \lim\limits_{n\to\infty} a_{nk} = \delta_k \quad (k = 0,1,2,\dots), \\[2ex] \text{(ii)} & \lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = \delta \,, \\[2ex] \text{(iii)} & \sum\limits_{k=0}^{\infty} |\, a_{n,k} \,| \le K \quad (n = 0,1,2,\dots\,). \end{cases}$$

In particular if, $\delta_k = 0$, for each k, and $\delta = 1$, then (1.3.1) gives the necessary and sufficient conditions for the transformation (1.2.1) to be T.

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§ Concerning such conditions for functional transformations reference may be made to TATCHELL [99].

The necessary and sufficient conditions for the transformation (1.2.1) to be AK, are (MEARS [59]) :

$$(1.3.2)\begin{cases}(i) & \sum_{k=0}^{\infty} a_{nk} \text{ converges, for each } n,\\ (ii) & \sum_{k=0}^{\infty} \left|\sum_{k=m}^{\infty} (a_{n,k}-a_{n-1,k})\right| \leq K (m=0,1,2,\ldots)\end{cases}$$

(1.3.2) implies the existence of the limits (i) and (ii) of (1.3.1).

In particular, the transformation (1.2.1) is AT, if $\delta_k = 0$, $\delta = 1$ ($k = 0,1,2,\ldots$) in (1.3.1) (i) and (ii).

Similarly, the necessary and sufficient conditions for the ϕ-transformations (1.2.2) and (1.2.3) to be T (or AT) are also known (Cf. [40], [13], [99] , see also [6], [7], [23], and [61]).

1.4. If, for any two methods of summability (or absolute summability), P and Q, we have $P \subseteq Q$, but $Q \not\subseteq P$, then the following questions can be raised ;

(a) Would it be possible in some manner to restrict the order of magnitude of the terms of the series Σa_n, so that, for it $Q \subseteq P$ (and in effect $P \sim Q$) ?

(b) Would there be sequences, $\{\epsilon_n\}$ such that $\Sigma \epsilon_n a_n$ may be summable P whenever Σa_n is summable Q ?

The result answering the question (a), in the affirmative is called 'Tauberian'. A result of the type : $P \subseteq Q$, or $P \subset Q$ is called 'Abelian'. The result answering the question (b) in the affirmative is called 'Summability factor-theorem' (or Absolute Summability factor theorem), and the sequence $\{\epsilon_n\}$ is then called the 'Summability (or Absolute Summability) Factors'.

In the present thesis we propose to investigate the problems concerning the results of the type mentioned above, and also study the applications of some methods of summability to Power series, Fourier series and its derived series.

1.5. SOME SPECIAL METHODS OF SUMMABILITY.

<u>Special |T|-methods</u>.

In special cases, the transform t_n of (1.2.1) reduces respectively to ;

(a) (C,α)-transform, s_n^{α}, $\alpha > -1$; when

$$a_{nk} = \begin{cases} A_{n-k}^{\alpha-1} / A_n^{\alpha} , & k \leq n, \\ 0 , & k > n ; \end{cases} \qquad \text{(cf. [40], p.96)}$$

(b) $(\bar{N},q_n)$-transform, $\bar{t}_n^q$: when

$$a_{nk} = \begin{cases} q_k/Q_n , & k \leq n , \\ 0 & k > n ; \end{cases} \quad (Q_n \neq 0), \text{ (cf. [40], p.)}$$

where $\{q_n\}$ is a sequence of constants, real or complex, such that

$$Q_n = q_0 + q_1 + \dots + q_n \to \infty, \text{ as } n \to \infty.$$

$\bar{t}_n^q = s_n^1$, whenever $q_n = 1$, for each $n = 0,1,2,\dots,$

(c) (N,p_n)-transform, t_n^p : when

$$a_{nk} = \begin{cases} p_{n-k}/P_n , & k \leq n , \ (P_n \neq 0), \\ 0 & , \ k > n ; \end{cases} \qquad \text{(cf. [106], [68])}$$

where $\{p_n\}$ is a sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + \dots + p_n ,$$

$t_n^p = \sigma_n^\alpha$, whenever $p_n = A_n^{\alpha-1}$, for each $n = 0,1,2,\dots$.

If $p_n = (n+1)^{-1}$, $P_n \sim \log n$, then the Nörlund mean reduces to $(N,(n+1)^{-1})$, or the harmonic mean ([39], see also [40]).

(d) (N,p_n,q_n)-transform, t_n^{p*q} : when

$$a_{nk} = \begin{cases} p_{n-k}\, q_k/r_n , & k \leq n \quad (r_n \neq 0), \\ 0 & , k < n, \end{cases}$$

where $\{p_n\}$ and $\{q_n\}$ are two fixed sequences, such that

$$r_n = p_n q_0 + p_{n-1} q_1 + \dots + p_0 q_n .$$

$t_n^{p*q} = \bar{t}_n^{\,q}$, when $p_n = 1$, for all $n \geq 0$, and $t_n^{p*q} = t_n^p$, when $q_n = 1$, for all $n \geq 0$, (cf. BORWEIN [15]).

The absolute summability methods associated with the above transformations are respectively the absolute Cesàro method of order $\alpha(\alpha > -1)$, absolute weighted arithmetic mean method, absolute Nörlund method and absolute generalised Nörlund method, and are denoted by : $|C,\alpha|$, $|\bar{N},q_n|$, $|N,p_n|$ and $|N,p_n,q_n|$ respectively, and the corresponding index methods by : $|C,\alpha|_k$, $|\bar{N},q_n|_k$, $|N,p_n|_k$ and $|N,p_n,q_n|_k$ respectively. It is evident that $|C,0|$ is the same as absolute convergence.

Special (ϕ) and $|\phi|$-methods.

In the special case in which

$$\phi_n(x) = p_n x^n / \left(\sum_{n=0}^{\infty} p_n x^n \right), \quad 0 \leq x < 1,$$

the transform $t(x)$ of (1.2.2) reduces to the (J,p_n)-transform, $J(x)$, defined by :

$$(1.5.1) \qquad J(x) = \left(\Sigma\, p_n x^n \right)^{-1} \Sigma\, p_n s_n x^n ,$$

where $p_n > 0$ $(n \geq 0)$, $\Sigma\, p_n = \infty$, and the series $\Sigma\, p_n x^n$ has radius of convergence 1, ([13]; see also [40], p.79-80). Then the associated (ϕ), $|\phi|$ and $|\phi|_k$-methods are denoted by:

(J,p_n), $|J,p_n|$ and $|J,p_n|_k$ respectively. $|J,p_n|_1$ is the same as $|J,p_n|$, and for $k > 1$, $|J,p_n|$ and $|J,p_n|_k$ are mutually independent (Cf. MAZHAR [56]).

In the special cases in which

(i) $p_n = A_n^{\alpha}$ $(\alpha > -1)$, (BORWEIN [12]),

(ii) $p_n = (n+1)^{-1}$ $(n=0,1,2,...)$ (BORWEIN [14], see also HARDY [40], p. 81),

(J,p_n), $|J,p_n|$ and $|J,p_n|_k$ reduce respectively to (A_α), (L) (BORWEIN [12], [14]), $|A_\alpha|$, $|L|$ (AHMAD [5], [7], DAS [23]), and $|A_\alpha|_k$ (MISRA [61]), $|L|_k$ (MAZHAR [56]). The methods (A_0), $|A_0|$, $|A_0|_k$ are Abel methods (A), $|A|$ and $|A|_k$ (Cf. [40], [105], [57]).

The earliest definition of any special method of absolute summability was that of $|C,\alpha|$-summability. Although it was introduced by FEKETE ([33], [34]) for non-negative integral order, it was studied by KOGBETLIANTZ [51] in considerable details, who proved that :

(KB-1) : $|C,\alpha| \subset (C,\beta)$, for every $\beta > \alpha > -1$;

(KE-2) : $|C,\alpha| \not\subseteq |C,\beta|$, for $\beta < \alpha$;

(KE-3) : $(C,\alpha) \not\subseteq |C,\beta|$, for $\beta > \alpha$.

The result (KE-1) has been obtained by a shorter method by MORLEY [65], and is also a particular case of a theorem of OBRECHKOFF [69].

The method $(\bar{N},q_n)$ is both T and AT, (See HARDY [40], p.57, and SUNOUCHI [96]). For regular $(\bar{N},q_n)$-method , SUNOUCHI [96] proved that :

(SU-1) : If $p_{n+1}/P_n > q_{n+1}/Q_n$, then $|\bar{N},p_n| \subset |\bar{N},q_n|$,

while PEYERIMHOFF [81] proved an elegant limitation theorem;

(PA-1): If $q_{n+1}/Q_{n+1} = O(q_n/Q_n)$, then $\Sigma \frac{q_n}{Q_n} |a_n| < \infty$,

whenever Σa_n is summable $|\bar{N},q_n|$.

Nörlund summability in its present form was introduced by NÖRLUND [68] in 1919. A definition substantially the same as that of Nörlund was given by WORONOI [106] in the proceedings of the eleventh congress of Russian naturalist and scientists, St. Petersburg,1902, 60-61 (in Russian). It was remained unnoticed till the first English translation of this work of Woronoi with remarks of translator by J.D.Tamarkin, was published in 1932 (see WORONOI [106]). In 1930, as a generalisation of Cesàro method, OBRECHKOFF [70] gave the same definition, quite independently of the works of Woronoi and Nörlund. He gave the regularity condition for this method :

(WGP-1): The necessary and sufficient conditions for the method (N,p_n) should be T, are : (i) $p_n = O(|P_n|)$, and (ii) $\sum_{v=0}^{n} |p_v| = O(|P_n|)$.

In 1937, MEARS [59] developed the concept of absolute Nörlund summability and gave the following result :

(MFM-1): Necessary and sufficient conditions for the method (N,p_n) should be AT, are (i) and (ii) above and

(iii) $\sum_{n=v}^{\infty} \left| \frac{P_{n-v}}{P_n} - \frac{P_{n-v-1}}{P_{n-1}} \right| < \infty$, for each v.

McFADDEN [57] discussed absolute Nörlund summability in considerable details and gave a number of results on the inclusion for the two Nörlund methods $|N,p_n|$ and $|N,q_n|$

and its applications to Fourier series. More results on this method have been established by DAS ([24], [25]), KISHORE ([47], [47a]), and others.

The absolute harmonic summability is due to McFadden who proved that :

(ML-1): $|N,(n+1)^{-1}| \subset |C,\alpha|$, for every $\alpha > 0$.

In view of the conditions for regularity and absolute regularity of Nörlund methods, harmonic method is both T and AT.

The general Nörlund method was defined and discussed by BORWEIN [15]. KUTTNER [53] has also studied some problems concerning this method. DAS [23] and others also discussed some aspect of the general absolute Nörlund summability method.

Now we come to the discussion of $|\phi|$-methods. Like Cesàro method Abel method is the most fundamental. By definition it is evident that $|A| \subset (A)$. Analogous to the

Abel's classical theorem, we also have the result :

(WJM-1): $|C,0| \subseteq |A|$, (see WHITTAKER [105]).

FEKETE [36] generalised this and proved the result :

(FM-1): $|C,\alpha| \subseteq |A|$, however large $\alpha (> 0)$ may be , and also showed by means of a negative example that :

(FM-2) : $|A| \not\subseteq (C,\alpha)$, and hence $|A| \not\subseteq |C,\alpha|$, however large $\alpha (> 0)$ may be.

This last result was also verified for Fourier series by RANDELS [88]. It has been demonstrated by PATI [71] that, for the conjugate series of a Fourier series, summability $|A|$ at a point, even when combined with everywhere convergence, does not necessarily imply summability $|C,1|$ at that point.

On the discovery of the fact that Dini's convergence criterion for Fourier series at a point is sufficient to ensure its summability $|A|$, WHITTAKER [105] was led to the consideration of the inter-relation between summability (C,0), i.e. convergence, and summability $|A|$. Using an example of Littlewood, he proved that :

(WJM-2) : $(C,o) \not\subseteq |A|$.

PRASAD [84], on the other hand proved that :

(PRN-1) : $|A| \not\subseteq (C,o)$.

HYSLOP [42] has proved the Tauberian theorem :

(HJM-1) : If, for the series Σa_n, $\Sigma \Delta(n a_n)$ is summable $|C,\alpha+1|$, then $|A| \subseteq |C,\alpha|$, for $\alpha \geq o$;

bad phrasing

In particular he proved that :

(HJM-2) : If for the series Σa_n, $\{n a_n\} \in BV$, then $|A| \subseteq |C,o|$.

FLETT [37] defined and discussed $|A_k|$-method in details and proved a number of results on the relation between the methods $|C,\alpha|_k$ and $|A|_k$. MAZHAR [55] and FLETT [38] independently proved the Tauberian theorem :

(MFM-1) : If Σa_n is summable $|A|_k$, $k \geq 1$, and $\Sigma \Delta(n a_n)$ is summable $|C,\alpha+1|_k$, where $\alpha > -1$, then Σa_n is summable $|C,\alpha|$.

Recently, AHMAD ([5], [6], [7]) and DAS ([23], [25]) have discussed a number of problems for $|J,p_n|$-family of methods. Ahmad has proved that :

(AZU-1): $|A| \subset |L|$ (See also DAS [23], MOHANTY and PATNAIK [64]).

(AZU-2): $|A_\alpha| \subseteq |A_\beta|$, for $\alpha > \beta \geq -1$ (See also MISRA [61]).

(AZU-3): (J,p_n) is AT, whenever $\Sigma\, p_n = +\infty$ (See also DAS [23]).

(AZU-4): $|\bar{N},q_n| \subset |J,q_n|$ (See also DAS [23], who proved that : $|N,p_n,q_n| \subseteq |J,q_n|$). In particular, we have

(AZU-5): $|\ell| \subset |L|$.*

Tauberian theorems have been discussed by DAS [25] for $|A_\lambda|$-summability, by AHMAD and RAHIMAN ([9], see also [86]), and AHMAD and VARSHNEY [10] for $|J,p_n|$-summability, and by RIZVI [90] for $|J,p_n|_k$-summability.

1.6. ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES.

(a) As early as in 1917, FEKETE [35] proved the theorem:

(FM-3): The necessary and sufficient conditions to be

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* 'logarithmic mean transform', ℓ_n, is defined by: $\ell_0 = s_0$, $\ell_1 = s_1$, $\ell_n = (\log n)^{-1}(s_0 + \frac{s_1}{2} + \ldots + \frac{s_n}{n+1})$, $n = 2, 3, \ldots$.

satisfied by a sequence $\{\epsilon_n\}$ such that $\Sigma\, \epsilon_n a_n$ is summable $|C,\alpha|$, whenever $\Sigma\, a_n$ is summable $|C,\alpha|$, are :

$$(i)\ \ \epsilon_n = O(1), \text{ and } (ii)\ \ n^{\alpha} \Delta^{\alpha}\epsilon_n = O(1),$$

where α is a non-negative integer.

The sufficiency part of this theorem was generalised by KOGBETLIANTZ [51] who proved that :

(KB-4): For $0 \leq \beta < \alpha$, $\Sigma\, a_n / n^{\alpha-\beta}$ is summable $|C,\beta|$ whenever $\Sigma\, a_n$ is summable $|C,\alpha|$. A subsequent extension of Kogbetliantz's theorem (KB-4) was due to SUNOUCHI [95] who could extend the range of β from $\beta \geq 0$ to $\beta > -1$.

After these pioneering works, the credit of discussing absolute Cesaro summability factors goes mainly to BOSANQUET [17], PEYERIMHOFF ([81], [82], see also [83]), CHOW [21], BOSANQUET and CHOW [18], PATI and AHMAD ([77], [78], [79]), and AHMAD ([2], see also [3], [5]). In 1945, BOSANQUET [17] established the theorem :

(BLS-1): The series $\Sigma\, \epsilon_n a_n$ is summable $|C,\beta|$, whenever

Σa_n is summable $|C,\alpha|$, if, and only if

$$\text{(i)} \quad \epsilon_n = \begin{cases} O(n^{\beta-\alpha}), & 0 \leq \beta < \alpha, \\ O(1), & \beta > \alpha, \end{cases}$$

and

$$\text{(ii)} \quad n^{\alpha} \Delta^{\alpha} \epsilon_n = O(1),$$

where α and β are non-negative integers.

In 1954, both CHOW [21] and PEYERIMHOFF [82] showed that the result (BLS-1) is also true for all non-negative α and β, while ANDERSEN [11] independently stated its sufficiency part. Chow's result was a generalisation in another direction and, as such, contained the theorem of Peyerimhoff as a special case.

A generalisation of the above-mentioned result (KB-4) of Kogbetliantz in the case $\beta = 0$, to absolute Nörlund summability, was given by DAS [22] who proved the theorem:

(DG-1) : If $\{p_n\}$ be such that $\sum_{\nu=0}^{n} |p_\nu| = O(|P_n|)$, and $\Sigma |c_n| < \infty$*, then $\Sigma \epsilon_n a_n$ is absolutely convergent.

* $\{c_n\}$ is given by the identity $(\Sigma p_n x^n)^{-1} = \Sigma c_n x^n$.

whenever Σa_n is summable $|N,p_n|$, if, and only if, $P_n \epsilon_n = O(1)$.

An analogous result for $|\bar{N},q_n|$-summability is due to PEYERIMHOFF [81] who proved :

(PA-1) : If $\{q_n\}$ be such that $q_{n+1}/Q_{n+1} = O(q_n/Q_n)$, then $\Sigma \frac{q_n}{Q_n} | a_n| < \infty$, whenever Σa_n is summable $|\bar{N},q_n|$.

In Chapter III of the present thesis we have considered the absolute convergence factors for $|N,p_n,q_n|$-method, so as to get the result of Das, and a result parallel to that of Peyerimhoff as special cases.

(b) If a given infinite series Σa_n is not summable $|N,p_n,q_n|$, but the nth total variation of the (N,p_n,q_n)-mean is of certain order, say μ_n, i.e.

$$\sum_{k=1}^{n} | t_n^{p*q} - t_{n-1}^{p*q} | = O (\mu_n) \, ,$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, then the natural question arises as to what type of sequences of factors $\{\epsilon_n\}$ can be chosen, so that the series $\Sigma \epsilon_n a_n$ may be summable $|N,p_n,q_n|$?

In special cases this question was answered by a number of authors, viz., SUNOUCHI [97], G.D.DIKSHIT [27], AHMAD ([3],[7]), PATI [74] and others. We, however, mention the following results with which we are directly concerned here.

(PL-1): If Σa_n is bounded $[R, \log n, 1]$†, and $\{\lambda_n\}$ is convex sequence such that $\Sigma n^{-1} \lambda_n$ is convergent, then

(i) $\Sigma \lambda_n a_n$ is summable $|C,1|$, (PATI [74], Theorem 1);

(ii) $\Sigma n^{-1}(\log \overline{n+1}) \lambda_n a_n$ is summable $|N,(n+1)^{-1}|$,

(LAL [54], Theorem 2).

(SK-1): If Σa_n is summable $|C,1|$, and $\{\lambda_n\}$ is a convex sequence such that $\Sigma n^{-1}\lambda_n$ is convergent, then

(i) $\Sigma n^{-1}(\log \overline{n+1}) \lambda_n a_n$ is summable $|N,(n+1)^{-1}|$.

(SINGH [92 a])

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† Σa_n is said to be bounded $[R, \log n, 1]$, if
$$\sum_{\nu=1}^{n} \frac{|a_\nu|}{\nu} = O(\log n).$$

(ii) $\Sigma\, n^{-1} P_n \lambda_n a_n$ is summable $|N, p_n|$, (KISHORE [46]).

Recently, answering this question for $|N, p_n|$-summability in a general form, AHMAD [7] has generalised all these results and obtained the result :

(AZU-6) : Let $p_0 > 0$, $p_n \geq 0 (n = 1, 2, \ldots)$, let $\{p_n\}$ be non-increasing. If

$$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\mu_n),$$

where τ_n is the nth Cesàro mean of the sequence $\{n\, a_n\}$ and $\{\mu_n\}$ is a positive non-decreasing sequence, and if the sequence $\{\varepsilon_n\}$ is such that

(i) $\varepsilon_n \mu_n = O(1)$, $\quad n \Delta \mu_n = O(\mu_n)$,

(ii) $\Sigma\, n \mu_n \, |\Delta^2 \varepsilon_n| < \infty$,

then the series $\Sigma (n+1)^{-1} P_n \varepsilon_n a_n$ is summable $|N, p_n|$.

In Chapter II, as Theorem 1, we generalise this result replacing the sequences $\{\varepsilon_n\}$ by a more general class of sequences.

For $|\bar{N},q_n|$-method KHAN [44] proved the following theorem :

(KFM-1): If $q_n > 0$ ($n = 0,1,2,\ldots$), $(n+1)q_n \leq K\, Q_n$,

$$\sum_{v=1}^{n} \frac{q_v}{Q_{v-1}} \,|\,\tilde{t}_v\,| = O(\mu_n),$$

where $\tilde{t}_n = (P_n)^{-1} \sum_{v=1}^{n} p_{v-1}\, a_v$, and $\{\mu_n\}$ is a positive monotonic non-decreasing sequence, and if the sequences $\{\epsilon_n\}$ and $\{\mu_n\}$ are such that,

$$\text{(i)} \quad \epsilon_n\, \mu_n = O(1),\ \Delta\mu_n = O\left(\frac{|\Delta q_n|}{q_n}\, \mu_n \right),$$

$$\text{(ii)} \quad \sum Q_n \,\left|\Delta\left(\frac{1}{q_n} \right)\right| \,\mu_n |\Delta \epsilon_{n+1}| < \infty,$$

$$\cancel{\text{(iii)} \quad \sum \frac{Q_n}{q_n}\, \mu_n \,|\,\Delta^2 \epsilon_n| < \infty},$$

then the series $\sum \epsilon_n a_n$ is summable $|\bar{N},q_n|$.

In Chapter IV, we prove a couple of analogous results for $|N,p_n,q_n|$-method, the second of which includes the above-mentioned results (AZU-6) and (KFM-1), while the first one is a necessary and sufficient type theorem for $|N,p_n,q_n|$-summability factors of infinite series.

1.7. ABSOLUTE SUMMABILITY FACTORS OF POWER SERIES AND FOURIER SERIES.

Let $f(t)$ be a periodic function, with period 2π, integrable in the sense of Lebesgue over $(-\pi,\pi)$, and let the Fourier series of $f(t)$ be :

$$(1.7.1) \qquad f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),$$

where

$$(1.7.2) \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \; (n = 1,2,\ldots) ,$$

and

$$(1.7.3) \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \; (n = 1,2,\ldots),$$

assuming as we may, without any loss of generality, that the constant term is zero. The Fourier series of $f(t)$, at $t = x$, is $\Sigma A_n(x)$ and will be denoted by $S\,[f]_x$. We also write

$$\phi(t) = \frac{1}{2} \left\{ f(x+t) + f(x-t) \right\} .$$

(a) We know that, as consequence of the applications to power series and Fourier series of the results of Section 1.6(b), their authors obtained the following :

(RL-1): If $f(z) = \Sigma\, c_n z^n$ is a power series of the complex class L, such that

(1.7.4) $\int_0^t |f(e^{i\theta})|\, d\theta = O(|t|)$, as $t \to +0$,

and $\{\lambda_n\}$ is a convex sequence such that $\Sigma\, n^{-1}\lambda_n$ is convergent, then

(i) $\Sigma\, \lambda_n c_n$ is summable $|C,1|$ (RAJAGOPAL [87], Theorem I),

(ii) $\Sigma\, n^{-1}(\log \overline{n+1})\lambda_n\, c_n$ is summable $|N,(n+1)^{-1}|$,
(LAL [54], Theorem 1).

(CL-1): If $\{\lambda_n\}$ is a convex sequence such that $\Sigma\, n^{-1}\lambda_n$ is convergent, then

(i) the series $\Sigma\, \lambda_n A_n(x)$ is summable $|C,1|$ for almost all values of x, (CHOW [20]).

(ii) the series $\Sigma\, n^{-1}(\log \overline{n+1})\lambda_n\, A_n(x)$ is summable $|N,(n+1)^{-1}|$ for almost all values of x, (LAL [54], Theorem 3).

(RCF-1): If F(x) is even, $F(x) \in L^2(-\pi,\pi)$,

(1.7.5) $\int_0^t |F(x)|^2\, dx = O(t)$, as $t \to +0$, and if $\{\lambda_n\}$ is a convex sequence such that $\Sigma\, n^{-1}\lambda_n$ is convergent, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\Sigma\, \lambda_n A_n$ is summable $|C,1|$, (RAJAGOPAL [87], Theorem II).

(RCT-2): If $F(x)$ is even, $F(x) \in L(-\pi,\pi)$,

(1.7.6) $\int_0^t |F(x)|\, dx = O(t)$, as $t \to +0$,

and if $\{\lambda_n\}$ is a convex sequence such that $\Sigma\, n^{-1}\lambda_n$ is convergent, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\Sigma\, (\log \overline{n+1})^{-1/2}\, \lambda_n A_n$ is summable $|C,1|$.

(AHU-7) : Let $p_0 > 0$, $p_n \geq 0$ $(n = 1,2,\ldots)$, and let $\{p_n\}$ be non-increasing. If $\{\epsilon_n\}$ is such that

(i) $\log n\, \epsilon_n = O(1)$,

(ii) $\Sigma\, n \log n\, |\Delta^2 \epsilon_n| < \infty$,

then the series $\Sigma (n+1)^{-1} P_n\, \epsilon_n\, A_n(x)$ is summable $|N,p_n|$, for almost all values of x, (AHMAD [7], Theorem 2).

(AZU-8) : Let the sequences $\{p_n\}$ and $\{\epsilon_n\}$ be the same as in (AZU-7). If $f(x)$ satisfies the conditions of (PL-1), then $\Sigma\ (n+1)^{-1}\ p_n\ \epsilon_n\ c_n$ is summable $|N,p_n|$. (AHMAD [7], Theorem 5).

(AZU-9) : Let the sequences $\{p_n\}$ and $\{\epsilon_n\}$ be the same as in (AZU-7). If $F(x)$ satisfies the conditions of (RCT-1), then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\Sigma(n+1)^{-1}\ p_n \epsilon_n\ A_n$ is summable $|N,p_n|$. (AHMAD [7], Theorem 3).

(AZU-10) : Let the sequences $\{p_n\}$ and $\{\epsilon_n\}$ be the same as in (AZU-7). If $F(x)$ satisfies the conditions of (AZU-7), then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\Sigma(n+1)^{-1}\ (\log \overline{n+1})^{-1/2}\ p_n\ \epsilon_n\ A_n$ is summable $|N,p_n|$.

In Chapter II, as applications of our Theorem 2.1, we obtain results which are the generalisations of results (AZU-7), (AZU-8) and (AZU-9) mentioned above.

(b) In 1936, BOSANQUET [16] proved that :

(BIS-2) : If $\phi(t) \in BV(0,\pi)$, then $S[f]_x$ is summable $|0,\delta|$, for every $\delta > 0$.

Generalising this result PATI ([72], [73a]) obtained the first theorem on the $|N,p_n|$-summability of $S[f]_x$ when $\phi(t) \in BV(0,\pi)$. Combined with subsequent work of his [75] Pati's result amounts to :

(PT-1): If $\phi(t) \in BV(0,\pi)$, and $\{p_n\}$ is a positive sequence, such that

$$\{R_n\} \equiv \left\{\frac{(n+1)p_n}{P_n}\right\} \in BV \text{ and } \{S_n\} \equiv \left\{\frac{1}{P_n} \sum_{k=0}^{n} \frac{P_k}{k+1}\right\} \in BV,$$

then $S[f]_x$ is summable $|N,p_n|$.

It has been observed by H.P. DIKSHIT [3] that in the statement of this result '$\{S_n\} \in BV$' is equivalent to '$\{S_n\} \in B$'. O.P. VARSHNEY [103] proved a theorem, equivalent to the above mentioned result of Pati. This result was further discussed extensively by a number of authors, viz., H.P. DIKSHIT ([30], [32]), WANG [104], MISRA [60], and G.D. DIKSHIT [28].

Obtaining a different kind of generalisation of the result (BLS-2) of Bosanquet, SINGH [92] proved the theorem:

(ST-1): If $\phi(t) \in BV(0,\pi)$ and $\{p_n\}$ be a non-negative and non-increasing sequence such that $\{\Delta p_n\}$ is non-increasing and $\{S_n\} \in B$, then $S[f]_x$ is summable $|N,p_n|$.

This result was later on improved upon by H.P.DIKSHIT [31], while KISHOR AND HOTTA [48] generalised it for absolute matrix summability.

On the other hand, since $\phi(t) \in BV(0,\pi)$, is not sufficient to ensure $|N,(n+1)^{-1}|$-summability of $S[f]_x$, (See PATI [73]), O.P VARSHNEY [102] proved that

(VOP-1) : If $\phi(t) \in BV(0,\pi)$, then Σ $(\log n)^{-1}$ $A_n(x)$, is summable $|N,(n+1)^{-1}|$.

Generalising Bosanquet's result : (BLS-2) in another direction SINGH [93] obtained a theorem for $|N,p_n|$-summability factors, so as to include O.P.Varshney's result a particular case when $p_n = (n+1)^{-1}$. He proved :

(ST-Q) : If $\phi(t) \in BV(0,\pi)$, then $\Sigma \frac{(n+1)p_n}{P_n} A_n(x)$ is summable $|N,p_n|$, where the sequence $\{p_n\}$ is real, non-negative and non-increasing such that

(i) $\left\{\frac{(n+1)p_n}{P_n}\right\} \in BV$,

(ii) $\{\Delta p_n\}$ is non-increasing.

Recently KANNO [43] gave a generalisation of this second result of Singh avoiding the condition (i). In Chapter V we generalise Konno's result in the same manner as KISHORE and HOTTA [40] generalised the first result of Singh.

1.8. TAUBERIAN THEOREMS FOR $|J,p_n|_k$-SUMMABILITY.

As a converse of the Abelian result : $|C,\alpha| \subset |A|$, in 1937, HYSLOP [42] established the following Tauberian theorem :

(HYN-1): If Σa_n is summable $|A|$ and $\Sigma \Delta(n a_n)$ is summable $|C,\alpha+1|$,‡ where $\alpha \geq 0$, then Σa_n is summable $|C,\alpha|$.

‡ It should be noted that the summability $|C,\alpha+1|$ of $\Sigma \Delta(n a_n)$ is also necessary for the summability $|C,\alpha|$ of Σa_n.

Particular cases of this theorem are :

(MJM-2) : If $\Sigma\, a_n$ is summable $|A|$ and the sequence $\{n\, a_n\}$ is summable $|C,1|$, then $\Sigma\, a_n$ is absolutely convergent.

(MJM-3) : If $\Sigma\, a_n$ is summable $|A|$, and if $\{n\, a_n\} \subset BV$, then $\Sigma\, a_n$ is absolutely convergent.

The last result may be regarded as the direct analogue for absolute summability of Tauber's second theorem.

(MJM-1) was generalised by [illegible] ([55], see also [illegible] [30]) by introducing the extended definition of absolute Abel and Cesàro summability referred as summability methods $|A|_k$ and $|C,\alpha|_k$. He proved :

(MSM-1) : If $\Sigma\, a_n$ is summable $|A_k|$, $k \geq 1$, and $\Sigma\, \Delta(n\, a_n)$ is summable $|C,\alpha+1|_k$, where $\alpha > -1$, then $\Sigma\, a_n$ is summable $|C,\alpha|$.

DAS ([26], Theorem 5) generalized the result :(MJM-1) in the following form :

(DG-2) : Let $k \geq 1$, $\beta > -1$, and $\alpha+\beta > -1$. If $\{s_n\}$ is summable $|A_\beta|_k$, then $\{s_n\}$ is summable $|C,\alpha,\beta|_k$ if, and only if $n\,a_n$ is summable $|C,\alpha+1,\beta|_k$.†

In view of the Abelian result $|\bar{N},p_n| \subset |J,p_n|$, (Cf. AHMAD [7], DAS [23]), AHMAD and RAHMAN ([9], see also RAHMAN [86]) attempted and proved the following Tauberian theorems for $|J,p_n|$-summability method.

(AR-1): If $\Sigma\, a_n$ is summable $|J,p_n|$ and $\{\tilde{t}_n\} \in BV$, and if $\{p_n\}$ is such that

(i) $\quad n\,p_n/P_{n-1} < C$, for $n = 1,2,\ldots$,

and

(ii) $\quad M(w) > 0$, for $w \geq 1$,

then $\Sigma\, a_n$ is summable $|\bar{N},p_n|$, where C is a strictly positive constant, and

$$\tilde{t}_n = (P_n)^{-1} \sum_{v=1}^{n} P_{v-1}\, a_v, \quad t_0 = 0 .$$

---

† $|C,\alpha,\beta|_k$ is a special case of $|N,p_n,q_n|_k$-method, when $p_n = A_n^{\alpha-1}$, $q_n = A_n^{\beta-1}$.

$$M(w) = \frac{\sum\limits_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v) p_u P_{v-u}}{\sum\limits_{v=0}^{\infty} e^{-v/w} \sum\limits_{u=0}^{v} (v-u+1) p_u P_{v-u}} .$$

(AR-2) : If $\sum a_n$ is summable $|J,p_n|$, and $\{\tilde{t}_n\} \in BV$, and if $\{p_n\}$ satisfies the same conditions as in (AR-1), then $\sum a_n$ is absolutely convergent.

(AR-3) : If $\sum a_n$ is summable $|J,p_n|$, and $\left\{\frac{P_{n-1}a_n}{p_n}\right\} \in BV$, and if $\{p_n\}$ satisfies the same conditions as in (AR-1), then $\sum a_n$ is absolutely convergent.

Thus (BMM-2) and (BMM-3) become special cases of (AR-2) and (AR-3) respectively.

Later on RIZVI [30] generalised these results: (AR-1)-(AR-3) by proving the following theorems for $|J,p_n|_k$-method.

(RBM-1): If, for $k \geq 1$, $\sum a_n$ is summable $|J,p_n|_k$, and $\{\tilde{t}_n\} \in BV^k$, and if $\{p_n\}$ is such that

(i) $\quad \frac{np_n}{P_{n-1}} < C$, for $n = 1,2,\ldots$

$$\text{(ii)} \quad \frac{M(w)^{k}}{w^{k-1}} > C , \text{ for } w \geq 1 ,$$

and

$$\text{(iii)}^{*} \quad \left(\frac{\sum_{v=0}^{\infty} p_v e^{-v/3w}}{\sum_{v=0}^{\infty} p_v e^{-v/w}} \right)^{k-1} \leq B, \text{ for } w \geq 1 ,$$

then Σa_n is summable $|M, p_n|_k$.

(RSM-2) : If, for $k \geq 1$, Σa_n is summable $|J, p_n|_k$, and $\{\tilde{t}_n\} \in BV^k$, and if $\{p_n\}$ satisfies the condition of (RSM-1), then Σa_n is summable $|C, o|_k$.†

(RSM-3) : If, for $k \geq 1$, Σa_n is summable $|J, p_n|_k$, and $\left\{ a_n \frac{P_{n-1}}{p_n} \right\} \in BV^k$, and if $\{p_n\}$ satisfies the condition (RSM-1), and if in addition,

$$\text{(iv)} \quad \left(\frac{1}{P_n} \sum_{v=1}^{n} \frac{P_v}{v} \right)^{k-1} \text{ is bounded for } k \geq 1, \text{ then}$$

Σa_n is summable $|C, O|_k$.

Since (AR-1) - (AR-3) cover only the Abel case (when

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* This condition is void for $k = 1$.

† Summability $|C, o|_k$ of Σa_n is the same as $\{a_n\} \in BV^k$.

$p_n = 1$ for all $n \geq o$) and do not cover the important case of logarithmic summability L (when $p_n = (n+1)^{-1}$, for all $n \geq o$), recently AHMAD and VARSHNEY ([10], see also K.O. VARSHNEY [101]) have proved the following :

(AV-1) : If Σa_n is summable $|J, p_n|$ and $\{\tilde{t}_n\} \in BV$, and if $\{p_n\}$ is such that

(i) $\quad \dfrac{\Delta p_n}{p_{n-1}} < 0$, for $n = 1,2,\ldots$,

and

(ii) $\quad M(w) > C \dfrac{m\, p_m}{P_{m-1}}$, for $w \geq 1$, where $m = [w]$,

then Σa_n is summable $|\bar{N}, p_n|$.

(AV-2) : If Σa_n is summable $|J, p_n|$, and $\{\tilde{t}_n\} \in BV$, and $\{p_n\}$ satisfies the same conditions as in (AV-1), with the condition (ii) replaced by the condition

(ii)' uniformly in $n \geq r \geq 1$,

$$\frac{p_n}{P_{n-1}} = O\left(\frac{p_r}{P_{r-1}} \right) ,$$

then Σa_n is summable $|\bar{N}, p_n|$.

In particular, they obtained :

(AV-3) : If Σa_n is summable $|J,p_n|$, and $\{\tilde{t}_n\} \in BV$, and if p_n (> 0) is non-increasing, then Σa_n is summable $|\bar{N},p_n|$.

The following results have also been deduced from (AV-1).

(AV-4) : If Σa_n is summable $|J,p_n|$, and $\{\tilde{t}_n\} \in BV$, and if $\{p_n\}$ satisfies the same conditions as in (AV-1), then Σa_n is absolutely convergent.

(AV-5) : If Σa_n is summable $|J,p_n|$, and $\{a_n P_{n-1}/p_n\} \in BV$, and if $\{p_n\}$ satisfies the same conditions as in (AV-1), then Σa_n absolutely convergent.

We observe that the result (AV-1) covers both the cases of absolute Abel and absolute logarithmic summability methods, while (AV-2) gives simplified conditions which are easy to apply.

In Chapter VI we extend these results (AV-1) - (AV-5) to $|J,p_n|_k$-summability, improving upon the results proved by Rizvi.

1.9. (J,p_n)-SUMMABILITY OF DERIVED FOURIER SERIES.

As in Section 1.7, let $f(t)$ be Lebesgue-integrable in $(-\pi,\pi)$ and periodic with period 2π, and let the Fourier series of $f(t)$ be :

(1.9.1) $$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t).$$

Then the first derived Fourier series of (1.9.1) is

(1.9.2) $$\sum_{n=1}^{\infty} n\,(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} n\,B_n(t).$$

The Fourier series of $f(t)$, at $t = x_0$, is $\sum_{n=0}^{\infty} A_n(x_0)$, and the derived Fourier series, at $t = x_0$, is $\sum_{n=1}^{\infty} n\,B_n(x_0)$ and will be denoted by $S[f]_{x_0}$ and $S'[f]_{x_0}$ respectively.

We also write

$$\phi^*(t) = \frac{1}{2}\left\{f(x_0+t) + f(x_0-t) - 2S\right\}$$

$$\psi(t) = f(x_0+t) - f(x_0-t),$$

$$g(t) = \frac{\psi(t)}{4 \sin \frac{1}{2} t} - C\,,$$

$$\phi_1^*(t) = \frac{1}{t} \int_0^t \phi^*(u)\,du\,,$$

$$g_1(t) = \frac{1}{t} \int_0^t g(u)\, du ,$$

Where S and C are the functions of x.

For the first time HSIANG [41] applied the (L)-method to Fourier series and proved the following theorems.

(HFC-1): A necessary and sufficient condition for $S[f]_{x_0}$ to be summable (L) to the sum S, is that

$$\int_0^{\pi} \frac{\phi^*(t)}{t} \tan^{-1} \left(\frac{x \sin t}{1-x \cos t} \right) dt = o \left(|\log (1-x)| \right),$$

as $x \to 1-0$.

(HFC-2) : The (L)-summability of $S[f]_t$ is a local property of $f(t)$ near $t = x_0$.

(HFC-3) : If

(i) $\int_0^t |\phi^*(u)| du = o \left(t \log \frac{1}{t}\right), \ (t \to +0),$

(ii) $\int_t^{\delta} \frac{|\phi^*(u)|}{u} du = o \left(\log \frac{1}{t} \right), \ (t \to +0),$

as $t \to +0$, for any arbitrary δ, $0 < \delta < \pi$, then $S[f]_{x_0}$ is summable to S.

Improving the result (HFC-3) NANDA [66] proved the following:

(MMM-1) : If

$$\int_t^{\pi} \frac{\phi^*(u)}{u}\, du = o\left(\log \frac{1}{t}\right), \quad (t \to +0),$$

then $S[f]_{x_0}$ is summable (L) to S.

Concerning the (L)-summability of the derived Fourier series, the following theorem is due to MOHANTY and NANDA [63] who proved :

(MN-1) : If

$$\int_t^{\pi} \frac{|g(u)|}{u}\, du = o\left(\log \frac{1}{t}\right), \quad (t \to +0),$$

then $S'[f]_{x_0}$ is summable (L) to C.

Subsequently NANDA [66] improved upon theorem (MN-1) and the following theorem was obtained.

(MMM-2) : If

$$\int_t^{\pi} \frac{g(u)}{u}\, du = o\left(\log \frac{1}{t}\right), \quad (t \to +0),$$

then $S'[f]_{x_0}$ is summable (L) to C.

Later on NANDA and DAS [67] improved upon the results (NMM-1) and (NMM2) and proved :

(ND-1) : If

$$\phi(t) = \int_t^\pi \frac{\phi_1^*(u)}{u}\, du = o\left(\log \frac{1}{t}\right), \ (t \to +0),$$

then $\sigma[f]_{x_0}$ is summable (L) to S.

(ND-2) : If

$$g(t) = \int_t^\pi \frac{g_1(u)}{u}\, du = o\left(\log \frac{1}{t}\right), \ (t \to +0),$$

then $\sigma'[f]_{x_0}$ is summable (L) to S.

In a recent note KHAN [45] has generalized theorems (MFC-1) – (MFC-3) by proving corresponding theorems for (J, p_n)-summability. He proved :

(KFM-2) : A necessary and sufficient condition for $\sigma[f]_{x_0}$ to be summable (J, p_n) to the sum S, is that

$$\int_0^\delta \frac{\phi^*(t)}{t} \operatorname{Im} p(xe^{it})\, dt = o(p(x)),$$

for any arbitrary δ , $o < \delta < \pi$, as $x \to 1 - o$.

(KPN-3) : The (J,p_n)- summability of $\sigma[f]_t$ is a local property of $f(t)$ near $t = x_o$, i.e.

$$p_n(x) = \frac{2}{\pi} \int_o^{\delta} \frac{\phi^*(t)}{t} \operatorname{Im} p(x\, e^{it})\, dt + o(p(x)) ,$$

for any arbitrary δ, $o < \delta < \pi$, as $x \to 1 - o$.

(KPN-4) : Let the sequence $\{p_n\}$ be positive and decreasing steadily to zero, such that $\{n\, p_n\}$ is bounded. If

$$\text{(i)} \quad \int_o^t |\phi^*(u)|\, du = o(t\, p(1-t)), \ (t \to + o),$$

$$\text{(ii)} \quad \int_t^{\delta} \frac{|\phi^*(u)|}{u}\, du = o\, (p(1-t)), \ (t \to + o),$$

for any arbitrary δ, $o < \delta < \pi$, then $\sigma[f]_{x_o}$ is summable (J,p_n) to s.

Very recently, K.C.VAN SYNEY ([10], Chapter VII) has generalised theorem (ND-1) by proving the following theorem for (J,p_n)-summability :

(VKC-1) : Suppose that

(i) $\{n\, p_n\} \in BV$

(ii) there is an a, $0 < a < 1$, such that

$$(1-x)^{a}\, p(x) \downarrow, \text{ as } x \uparrow 1 .$$

If

$$\bar{\Phi}(t) = \int_t^{\pi} \frac{\phi_1(u)}{u}\, du = o(p(1-t)),\ (t \to +0),$$

then $[f]_{x_0}$ is summable (J, p_n) to 0.

In the manner similar to that of Theorems (KPM-3) and (VKC-1), in Chapter VII, we generalise theorem (ND-2) by obtaining a corresponding result for (J, p_n)-summability of derived Fourier series.

Chapter II

ON $|N,p_n|$-SUMMABILITY FACTORS OF INFINITE SERIES WITH APPLICATIONS

2.1 <u>Definitions and Notations</u> : Let $\Sigma\, a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \dots + p_n \; ; \; P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation :

$$(2.1.1) \qquad t_n^p = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v}\, s_v \quad (P_n \neq 0),$$

defines the sequence $\{t_n^p\}$ of Norlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\Sigma\, a_n$ is said to be summable (N,p_n), to the sum s, if $\lim\limits_{n \to \infty} t_n$ exists and is equal to s.

The series $\Sigma\, a_n$ is said to be absolutely summable (N,p_n), or summable $|N,p_n|$, if $\{t_n^p\} \in BV$, (MEARS [58]).

In the special case in which

$$(2.1.2) \quad p_n = A_n^{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,\Gamma(n+1)} \quad (\alpha > -1),$$

the Nörlund mean reduces to the familiar (C,α)-mean ([40], §5.13). The summability $|N,p_n|$, with p_n defined by (2.1.2), is thus the same as summability $|C,\alpha|$. Similarly, in the case in which

$$(2.1.3) \quad p_n = \frac{1}{(n+1)}, \; P_n \sim \log n, \text{ as } n \to \infty,$$

(N,p_n) reduces to the familiar harmonic mean (see RIESZ [89]). $|N,p_n|$-summability is then the same as absolute harmonic summability. It is well known that $|N,(n+1)^{-1}| \subset |C,\alpha|$ for every positive α, (McFADDEN [57]).

2.2. If

$$(2.2.1) \quad \sum_{\nu=1}^{n} |s_\nu| = O(n), \quad \text{as } n \to \infty,$$

the series $\Sigma\, a_n$ is said to be strongly bounded by Cesàro means of order 1, or bounded $[C,1]$. If

$$(2.2.2) \qquad \sum_{v=1}^{n} \frac{|s_v|}{v} = O(\log n), \text{ as } n \to \infty ,$$

the series $\sum a_n$ is said to be strongly bounded by 'logarithmeans' with index 1, or bounded $[R, \log n, 1]$, (see [74]).

Let s_n^1 and τ_n denote the n^{th} (C,1)-means of the sequences $\{s_n\}$ and $\{n\,a_n\}$ respectively, viz.

$$s_n^1 = \frac{1}{n+1} \sum_{v=0}^{n} s_v \; ; \; \tau_n = \frac{1}{n+1} \sum_{v=1}^{n} v\,a_v, \; (n=0,1,\ldots).$$

Then, by an identity of KOGBETLIANTZ [51], i.e., $n(s_n^1 - s_{n-1}^1) = \tau_n$, the n^{th} total variation of the sequence $\{s_n^1\}$ is given by :

$$(2.2.3) \qquad \sum_{v=1}^{n} | s_v^1 - s_{v-1}^1 | = \sum_{v=1}^{n} \frac{|\tau_v|}{v} .$$

2.3 The concept of quasi-convex sequence was recently generalised by TELYAKOVSKII [100] as follows :

A sequence $\{a_n\}$ is said to belong to class S if the following conditions are satisfied

(i) $a_n \to o$, $n \to \infty$,

(ii) there exists a sequence of numbers $\{\alpha_k\}$ such that $\alpha_k \downarrow 0$ and $\sum_{k=1}^{\infty} \alpha_k$ is convergent.

(iii) $|\Delta a_k| \leq \alpha_k$, for all k.

Taking $\alpha_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|$ it follows that a null quasi-convex sequence $\{a_n\}$ belongs to the class S. The converse is obviously not true. In view of the conditions (ii) and (iii), it follows that every sequence $\{a_n\}$ of class S is of bounded variation and that $n \Delta a_n \to o$, as $n \to \infty$.

A sequence $\{a_n\}$ of positive numbers is said to be quasi-monotone if $\Delta a_n \geq - \beta n^{-1} a_n$ for some positive β. It is obvious that every null-monotonic decreasing sequence is quasi-monotone. The sequence $\{a_n\}$ is said to be δ-quasi-monotone if $a_n \to o$, $a_n > o$ ultimately and $\Delta a_n \geq - \delta_{n+1}$; where $\{\delta_n\}$ is a sequence of positive numbers. It is easy to see that a null quasi-monotone sequence is δ-quasi-monotone with $\delta_{n+1} = \beta\, n^{-1} a_n$.

A sequence $\{a_n\}$ will be said to belong to class $S(\delta)$ if

(i) $a_n \to 0, \quad n \to \infty$,

(ii) there exists a sequence of numbers $\{\alpha_n\}$ such that it is δ-quasi-monotone with : $\sum_{n=1}^{\infty} \alpha_n$ is convergent,

(iii) $|\Delta a_n| \leq \alpha_n$, for all n.

It is trivial that $a_n \in S \Longrightarrow a_n \in S(\delta)$.

2.4 Let $f(t)$ be a periodic function, with period 2π, integrable in the sense of Lebesgue over $(-\pi,\pi)$, and let

$$(2.4.1) \quad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$\equiv \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t)$$

be the Fourier series of the function $f(t)$.

We write

$$P_{n,v} = P_n \, p_{n-v} - P_{n-v} \, p_n ,$$

for $v \geq 1$, $n \geq 1$.

2.5 Introduction : Generalising a number of results (See [46], [54], [74], [87], [92a]) AHMAD [8] proved the following theorems on the absolute Nörlund summability factors of power series and Fourier series.

Theorem A. Let $p_0 > 0$, $p_n \geq 0$ ($n = 1,2,\ldots$), and let $\{p_n\}$ be non-increasing. If

$$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\mu_n),$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, and if the sequence $\{\epsilon_n\}$ is such that

(i) $\epsilon_n \mu_n = O(1)$, $n \Delta \mu_n = O(\mu_n)$,

(ii) $\Sigma \, n \mu_n \, | \Delta^2 \epsilon_n | < \infty$,

then the series $\Sigma (n+1)^{-1} P_n \epsilon_n a_n$ is summable $|N, p_n|$.

<u>Theorem</u> B. Let p_n be the same as in Theorem A. If $\{\epsilon_n\}$ is such that

(i) $\log n \, \epsilon_n = O(1)$,

(ii) $\sum n \log n \, | \Delta^2 \epsilon_n | < \infty$,

then the series $\sum (n+1)^{-1} p_n \epsilon_n A_n(x)$ is summable $|N, p_n|$ for almost all values of x.

<u>Theorem</u> C. Let p_n be the same as in Theorem A. If $F(x)$ is even, $F(x) \in L^2(-\pi, \pi)$,

$$(2.5.1) \quad \int_0^t |F(x)|^2 \, dx = O(t) ,$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem B, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has a property that $\sum (n+1)^{-1} p_n \epsilon_n A_n$ is summable $|N, p_n|$.

<u>Theorem</u> D. Let p_n be the same as in Theorem A. If $F(x)$ is even, $F(x) \in L(-\pi, \pi)$,

$$\int_0^t |F(x)|\, dx = O(t),$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem B, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\Sigma(n+1)^{-1} (\log n)^{-1/2} P_n \epsilon_n A_n$ is summable $|N,p_n|$.

Theorem E. If $f(x) = \Sigma\, c_n z^n$ is a power series of the complex class L, such that

$$(2.5.2) \qquad \int_0^t |f(e^{i\theta})|\, d\theta = O(|t|),$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem B, then $\Sigma(n+1)^{-1} P_n \epsilon_n c_n$ is summable $|N,p_n|$.

Our object in this chapter is to establish a further generalisation of the above mentioned theorems, by replacing sequences $\{\epsilon_n\}$ by a sequence belonging to the class $S(\delta)$, so that these theorems become special cases of our theorems.

2.6 We prove the following theorems :

<u>Theorem</u> 2.1. <u>Let</u> $p_0 > 0$, $p_n \geq 0$ ($n = 1,2,\dots$) <u>and</u>

let $\{p_n\}$ be non-increasing. If

(2.6.1) $$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\mu_n),$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, and if the sequence $\{\epsilon_n\}$ is such that

(a) $\epsilon_n \mu_n = O(1)$, $n \Delta \mu_n = O(\mu_n)$,

(b) there exists a sequence of numbers $\{\alpha_k\}$ such that it is δ-quasi-monotone with :

$$\sum_{n=1}^{\infty} n \mu_n \delta_n < \infty ,$$

$$\Sigma \mu_n \alpha_n < \infty ,$$

and (c) $|\Delta \epsilon_n| \leq \alpha_n$, for all n ;

then the series $\Sigma(n+1)^{-1} P_n \epsilon_n a_n$ is summable $|N,p_n|$.

Theorem 2.2. Let p_n be the same as in Theorem 2.1. If the sequence $\{\epsilon_n\}$ is such that

(a) $\log n\, e_n = O(1)$,

(b) *there exists a sequence of numbers* $\{\alpha_k\}$ *such that it is* δ*-quasi-monotone with* :

$$\sum_{n=1}^{\infty} n \log n\, \delta_n < \infty ,$$

$$\Sigma \log n\, \alpha_n < \infty ,$$

and

(c) $|\triangle e_n| \leq$ for all n ;

then the series $(n+1)^{-1} p_n e_n A_n(x)$ *is summable* $|N,p_n|$ *for almost all values of* x.

Theorem 2.3. *Let* p_n *be the same as in the Theorem* 2.1. *If* $F(x)$ *is even*, $F(x) \in L^2(-\pi,\pi)$,

(2.6.2) $$\int_0^t |F(x)|^2\, dx = O(t) ,$$

as $t \to + 0$, *and if the sequence* $\{e_n\}$ *satisfies the same conditions as in Theorem* 2.2, *then the sequence* $\{A_n\}$ *of the Fourier coefficients of* $F(x)$ *has the property that the series* $\Sigma\, (n+1)^{-1} p_n e_n A_n$ *is summable* $|N,p_n|$.

Theorem 2.4. *Let* p_n *be the same as in the Theorem* 2.1. *If* $F(x)$ *is even,* $F(x) \in L(-\pi,\pi)$,

$$\int_0^t |F(x)|\, dx = O(t),$$

as $t \to +0$, *and if the sequence* $\{c_n\}$ *satisfies the same conditions as in Theorem* 2.2, *then the sequence* $\{A_n\}$ *of the Fourier coefficients of* $F(x)$ *has the property that the series* $\Sigma(n+1)^{-1} (\log n)^{-1/2}\, p_n\, c_n\, A_n$ *is summable* $|N,p_n|$.

Theorem 2.5. *If* $f(z) = \Sigma\, c_n\, z^n$ *is a power series of the complex class* L, *such that*

$$(2.6.3) \qquad \int_0^t |\, f(e^{i\theta})\, |\, d\theta = O(|t|),$$

as $t \to +0$, *and if the sequence* $\{\epsilon_n\}$ *satisfies the same conditions as in Theorem* 2.2, *then the series* $\Sigma(n+1)^{-1} p_n \epsilon_n c_n$ *is summable* $|N,p_n|$.

Remarks. It is to be observed that whenever we take $a_m = \sum_{n=m}^{\infty} \mu_n \,|\, \Delta^2 \epsilon_n|$, and $\Sigma\, n\, \mu_n |\Delta^2 \epsilon_n| < \infty$, the conditions (a), (b) and (c) of our Theorem 2.1 are automatically satisfied and hence Theorem A becomes a particular case of Theorem 2.1.

2.7 We need the following lemmas for the proof of our theorems.

Lemma 2.1 ([8], Lemma 2, see also [4]). Let $p_0 > 0$, $p_n \geq 0 (n = 1,2,\ldots)$, and let $\{p_n\}$ be non-increasing. Then, for $v \geq 1$,

$$\text{(a)} \quad \sum_{n=v}^{\infty} \frac{p_{n,v}}{P_n P_{n-1}} \leq K ,$$

$$\text{(b)} \quad \sum_{n=v}^{\infty} \frac{| \Delta_v p_{n,v} |}{P_n P_{n-1}} \leq \frac{K}{P_v} .$$

Lemma 2.2 ([91], Theorems 1 and 2). Let the sequence $\{a_n\}$ be δ-quasi-monotone such that $\sum \phi_n \delta_n < \infty$, $\{\phi_n\}$ being a positive monotonic increasing sequence. If $\sum a_n \Delta \phi_n$ converges, then

$$\text{(i)} \qquad a_n \phi_n = o(1), \quad \text{as } n \to \infty ,$$

and

(ii) $\sum_{n=0}^{\infty} \phi_{n+1} \, |\Delta a_n| < \infty$.

Lemma 2.3. Let the sequence $\{\alpha_n\}$ be δ-quasi-monotone sequence with $\Sigma \, n \, \mu_n \, \delta_n < \infty$ and $\Sigma \, \mu_n \, \alpha_n < \infty$, $\{\mu_n\}$ being a positive non-decreasing sequence, such that $n \, \Delta \mu_n = O(\mu_n)$, then

(i) $\quad n \, \mu_n \, \alpha_n = o(1), \; n \to \infty$,

and

(ii) $\quad \Sigma \, n \, \mu_n \, |\Delta \alpha_n| < \infty$.

Proof. The proof of the lemma is the same as given by VARSHNEY ([101], Chapter II). We give it here for completeness. Taking $a_n = \alpha_n$ and $\phi_n = n \, \mu_n$ in Lemma 2.2, we see that

$$\begin{aligned}\Sigma \, |a_n \, \Delta \phi_n| &= \Sigma \, |\alpha_n \, \Delta(n \, \mu_n)| \\ &= \Sigma \, \alpha_n \, (\mu_n + (n+1) \, |\Delta \mu_n|) \\ &\leq K \, \Sigma \, \alpha_n \, \mu_n \\ &< \infty,\end{aligned}$$

by hypotheses.

Lemma 2.4. *Under the hypothesis of Theorem* 2.1, *and taking* $\bar{\epsilon}_n = (n+1)^{-1} p_n \epsilon_n$, *we have*

(i) $$\sum_{v=1}^{\infty} \frac{|\bar{\epsilon}_v||\tau_v|}{P_v} \leq K,$$

(ii) $$\sum_{v=1}^{\infty} |\Delta\bar{\epsilon}_v||\tau_v| \leq K.$$

Proof. (i) we have, as $m \to \infty$,

$$\sum_{v=1}^{m} \frac{|\bar{\epsilon}_v|}{P_v} |\tau_v| = O\Big(\sum_{v=1}^{m} |\epsilon_v| \frac{|\tau_v|}{v}\Big)$$

$$= O\Big(\sum_{v=1}^{m-1} \mu_v |\Delta\epsilon_v|\Big) + O(|\epsilon_m| \mu_m)$$

$$= O\Big(\sum_{v=1}^{m-1} \mu_v a_v\Big) + O(|\epsilon_m| \mu_m)$$

$$= O(1),$$

by hypotheses.

(ii) Since

$$\Delta\bar{\epsilon}_v = \frac{p_n}{n+1} \Delta\epsilon_n - \frac{p_{n+1}}{n+1} \epsilon_{n+1} + \frac{P_{n+1}}{n+2} \frac{\epsilon_{n+1}}{n+1},$$

we have

$$| \Delta \bar{\epsilon}_n | = O(|\Delta \epsilon_n| \frac{P_n}{n}) + O(\frac{|\epsilon_{n+1}|}{n}).$$

Hence, as $m \to \infty$,

$$\sum_{v=1}^{m} |\Delta \bar{\epsilon}_v| |\zeta_v| = O(\sum_{v=1}^{m} P_v |\Delta \epsilon_v| \frac{|\zeta_v|}{v}) + O(\sum_{v=1}^{m} |\epsilon_{v+1}| \frac{|\zeta_v|}{v})$$

$$= O(\sum_{v=1}^{m} P_v \alpha_v \frac{|\zeta_v|}{v}) + O(\sum_{v=1}^{m} |\epsilon_{v+1}| \frac{|\zeta_v|}{v})$$

$$= O(\sum_{v=1}^{m-1} v \mu_v | \Delta \alpha_v |) + O(\sum_{v=1}^{m-1} \alpha_{v+1} \mu_v) + O(m \alpha_m \mu_m) +$$

$$+ O(|\epsilon_{m+1}| \mu_m)$$

$$= O(1),$$

by hypotheses.

<u>Lemma</u> 2.5 [87]. <u>If</u> $f(x) = \Sigma c_n z^n$ <u>is a power series of complex class</u> L, <u>such that</u>

$$\int_0^t |f(e^{i\theta}) | d\theta = O(|t|),$$

<u>as</u> $t \to + 0$, <u>then</u> Σc_n <u>is bounded</u> $[R, \log n, 1]$.

Lemma 2.6 ([74],p.294). *If* Σa_n *is bounded* $[R, \log n, 1]$, *then* $\sum_{\nu=1}^{n} \frac{|\tau_\nu|}{\nu} = O(\log n)$, *as* $n \to \infty$.

Lemma 2.7 ([8],Lemma 10). *Let*

$$\tau_n(x) = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu A_\nu(x).$$

Then

$$\sum_{\nu=1}^{n} \frac{|\tau_\nu(x)|}{\nu} = o(\log n),$$

as $n \to \infty$, *for almost all values of* x.

Lemma 2.8 ([8],Lemma 8). *If* Σa_n *is bounded* $[C,1]$, *then it is bounded* $[R, \log n, 1]$.

Lemma 2.9 ([87], Lemma 4). *Let* $F(x)$ *be even* , $F(x) \in L^2(-\pi,\pi)$, *and let* S_n *denote the* n^{th} *partial sum of its Fourier series at the origin. Then if*

$$\int_0^\theta |F(x)|^2 \, dx = O(\theta) ,$$

as $\theta \to +0$, S_n *will be summable* $[C,1]$.

Lemma 2.10 [87]. *Let* $F(x)$ *be even*, $F(x) \in L(-\pi,\pi)$, *and let* S_n *denote the* n^{th} *partial sum of its Fourier series at the origin. Then, if*

$$\int_0^\theta |F(x)|\,dx = O(\theta) \ ,$$

as $\theta \to +0$, *then*

$$\sum_{v=1}^{n} |S_v| = O\left\{n\,(\log n)^{1/2}\right\} .$$

2.8 *Proof of Theorem* 2.1. Let $\bar{\varepsilon}_n = (n+1)^{-1} P_n \varepsilon_n$, and let $t_n^{p^*}$ denote the n^{th} Nörlund mean of the series $\Sigma\, \bar{\varepsilon}_v a_v$. Then, by definition, we have

$$t_n^{p^*} = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} \sum_{\beta=0}^{v} \bar{\varepsilon}_\beta a_\beta = \frac{1}{P_n} \sum_{v=0}^{n} P_{n-v}\, \bar{\varepsilon}_v a_v$$

and

$$t_n^{p^*} - t_{n-1}^{p^*} = \frac{1}{P_n P_{n-1}} \sum_{v=1}^{n} (P_n\, P_{n-v} - P_{n-v}\, P_n)\, \bar{\varepsilon}_v a_v$$

$$= \frac{1}{P_n P_{n-1}} \sum_{v=1}^{n} P_{n,v}\, \bar{\varepsilon}_v a_v$$

$$= \frac{1}{P_n P_{n-1}} \sum_{v=1}^{n} p_{n,v} \frac{\bar{\varepsilon}_v}{v} \zeta_v + \frac{1}{P_n P_{n-1}} \sum p_{n,v} \Delta \bar{\varepsilon}_v \zeta_v +$$

$$+ \frac{1}{P_n P_{n-1}} \sum_{v=1}^{n} \Delta_v p_{n,v} \bar{\varepsilon}_{v+1} \zeta_v$$

$$= \frac{1}{P_n P_{n-1}} (\Sigma_1 + \Sigma_2 + \Sigma_3), \text{ say.}$$

Therefore, in order to prove that $\sum_n |t_n^{p^*} - t_{n-1}^{p^*}| \leq K$, it is enough to show that

$$\sum_n \frac{1}{P_n P_{n-1}} |\Sigma_r| \leq K \ (r = 1,2,3).$$

Now, we have

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} |\Sigma_1| \leq \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{v=1}^{n} p_{n,v} |\bar{\varepsilon}_v| \frac{|\zeta_v|}{v}$$

$$= \sum_{v=1}^{\infty} |\bar{\varepsilon}_v| \frac{|\zeta_v|}{v} \sum_{n=v}^{\infty} \frac{p_{n,v}}{P_n P_{n-1}}$$

$$\leq K \sum_{v=1}^{\infty} |\bar{\varepsilon}_v| \frac{|\zeta_v|}{v} \quad \text{(by Lemma 2.1(a))}$$

$$\leq K \sum_{v=1}^{\infty} \frac{|\bar{\varepsilon}_v|}{P_v} | \zeta_v|$$

$$\leq K .$$

by hypotheses and Lemma 2.4 (i).

Next, we have

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} |\Sigma_2| \leq \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{v=1}^{n} p_{n,v} |\Delta \varepsilon_v| |\tau_v|$$

$$= \sum_{v=1}^{\infty} |\Delta \varepsilon_v| |\tau_v| \sum_{n=v}^{\infty} \frac{p_{n,v}}{P_n P_{n-1}}$$

$$\leq K \sum_{v=1}^{\infty} |\Delta \varepsilon_v| |\tau_v| \quad \text{(by Lemma 2.1(a))}$$

$$\leq K ,$$

by hypotheses and Lemma 2.4 (ii).

Finally, we have

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} |\Sigma_3| \leq \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{v=1}^{n} |\Delta_v p_{n,v}| |\varepsilon_{v+1}| |\tau_v|$$

$$= \sum_{v=1}^{\infty} |\varepsilon_{v+1}| |\tau_v| \sum_{n=v}^{\infty} \frac{|\Delta_v \, p_{n,v}|}{P_n \, P_{n-1}}$$

$$\leq K \sum_{v=1}^{\infty} \frac{|\varepsilon_{v+1}| |\tau_v|}{P_v} \quad \text{(by Lemma 2.1(b))}$$

$$\leq K ,$$

by hypotheses and Lemma 2.4 (i).

This completes the proof of Theorem 2.1.

2.9 Proof of Theorems 2.2, 2.3, 2.4 and 2.5. We obtain Theorem 2.2 from Theorem 2.1, by taking $\lambda_n = \log n$, and by an appeal to Lemma 2.7.

Theorem 2.3 can be obtained from Theorem 2.1, by taking $\lambda_n = \log n$, and by successive applications of Lemmas 2.9, 2.8 and 2.6.

We get Theorem 2.4 from Theorem 2.1 with $\lambda_n = (\log n)^{3/2}$, and with $c_n/(\log n)^{1/2}$ in place of c_n, by an appeal to Lemma 2.10 and by using the fact that

$$\sum_{\nu=1}^{n} |\tau_\nu| = O\left\{n(\log n)^{1/2}\right\}$$

implies

$$\sum_{\nu=1}^{n} \frac{|\tau_\nu|}{\nu} = O\left\{(\log n)^{3/2}\right\}.$$

Finally, we obtain Theorem 2.5 from Theorem 2.1 with $\lambda_n = \log n$, by appealing to Lemmas 2.5 and 2.6.

Chapter III

A NOTE ON ABSOLUTE CONVERGENCE FACTORS

3.1 <u>Definitions and Notations</u> : Let Σa_n be a given infinite series with sequence of partial sums, $\{s_n\}$. Let $\{p_n\}$, $\{q_n\}$ be two fixed sequences. The generalised Nörlund transformation, or (N,p_n,q_n)-transform, which was considered by BORWEIN [15], is defined as the sequence-to-sequence transformation :

$$(3.1.1) \qquad t_n^{p*q} = \frac{1}{r_n} \sum_{v=o}^{n} \sum_{v=o}^{n} p_{n-v}\, q_v\, s_v ,$$

where

$$r_n = \sum_{v=o}^{n} p_{n-v}\, q_v, \qquad (r_n \neq o, \text{ for all } n).$$

The series Σa_n is said to be absolutely summable by the (N,p_n,q_n)-transform, or summable $|N,p_n,q_n|$, if $\{t_n^{p*q}\} \in BV$.

The (N,p_n,q_n)-transform (3.1.1) reduces respectively to :

(a) (N,p_n)-transform (when $q_n = 1$, for all n), given by:

$$t_n^{p} = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} a_\nu, \qquad (P_n \neq 0).$$

In particular, when $p_n = \binom{n+k}{n}$, $n = 0,1,2,\ldots,$ t_n is the same as (C,k)-transform :

$$s_n^k = \frac{1}{A_n^k} \sum_{\nu=0}^{n} A_{n-\nu}^{k-1} a_\nu .$$

(b) $(\bar{N},q_n)$-transform (when $p_n = 1$, for all n) given by :

$$\bar{t}_n^{q} = \frac{1}{Q_n} \sum_{\nu=0}^{n} q_\nu a_\nu , \qquad (Q_n \neq 0).$$

Then, in the cases (a) and (b) $|N,p_n,q_n|$ is the same as $|N,p_n|$(in particular $|C,k|$) and $|\bar{N},q_n|$ respectively.

The necessary and sufficient conditions for the regularity of (N,p_n)-transform, are :

(3.1.2) $$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 ,$$

and

(3.1.3) $$P_n^* = \sum_{\nu=0}^{n} |p_\nu| = O(|P_n|), \text{ as } n \to \infty ,$$

and that for $(\bar{N},q_n)$-transform is $\sigma_n \to \infty$, as $n \to \infty$.

3.2 Let us write

$$(3.2.1) \qquad p(x) = \sum_{n=0}^{\infty} p_n x^n$$

and let us define the sequence of constants $\{c_n\}$ by means of the identity :

$$(3.2.2) \qquad \left(\sum_{n=0}^{\infty} p_n x^n \right)^{-1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_{-1} = o,$$

whenever it holds. In particular, the relation (3.2.2) holds for $|x| < \delta$, for some $\delta > o$, whenever $\sum_{n=0}^{\infty} p_n x^n$ has a positive radius of convergence and $p_o = P_o \neq o$. In the case in which $p_n = A_n^{k-1}$, $k > -1$, $c_n = A_n^{-k-1}$.

We write $p_n \in \mathcal{M}$ to mean that $\{p_n\}$ belongs to the class of sequence $\mathcal{M}$ for which

$$(3.2.3) \qquad p_n > o, \; \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \quad (n = o,1,2,\ldots)$$

holds.

3.3 <u>Introduction</u> : KOGBETLIANTZ [51] generalised the sufficiency part of a result of FEKETE [35] and obtained the following.

<u>Theorem</u> A. The series $\sum a_n/n^{\alpha-\beta}$ is summable $|C,\beta|$ whenever $\sum a_n$ is summable $|C,\alpha|$, $0 \leq \beta < \alpha$.

For this theorem the range of β was extended to $\beta > -1$, by SUNOUCHI [95].

BOSANQUET ([17], Theorem 2) generalised all these results and established the following.

<u>Theorem</u> B. If $0 \leq \beta \leq \alpha$, (α,β integers), the necessary and sufficient conditions for $\sum a_n \epsilon_n$ to be summable $|C,\beta|$ whenever $\sum a_n$ is summable $|C,\alpha|$, are :

(i) $\epsilon_n = O(n^{\beta-\alpha})$, and (ii) $\Delta^\alpha \epsilon_n = O(n^{-\alpha})$.

If $\beta > \alpha \geq 0$, the conditions are the same as in the case of $\beta = \alpha$.

CHOW [21] and PEYERIMHOFF [82] showed that the above result also holds true for non-integral α, β.

DAS [22] has recently generalised Theorem A for the case $\beta = 0$, to absolute Nörlund summability and proved the following theorems :

Theorem C. Let $\{p_n\}$ be such that

$$\text{(i)} \quad \sum_{n=0}^{\infty} |c_n| < \infty \text{, and} \quad \text{(ii)} \quad P_n^* \leq K \, |P_n|.$$

Then, the necessary and sufficient condition that $\Sigma \, a_n \, \epsilon_n$ should be absolutely convergent whenever $\Sigma \, a_n$ is summable $|N, p_n|$, is $\epsilon_n \, P_n = O(1)$, as $n \to \infty$.

Theorem D. If in Theorem B the condition (ii) is dropped, the condition

$$\epsilon_n \quad P_n^* = O(1), \quad \text{as} \quad n \to \infty$$

is sufficient for the absolute convergence of $\Sigma \, a_n \, \epsilon_n$.

PEYERIMHOFF [81] obtained the following analogous result for $|\bar{N}, q_n|$-summability, which is a direct generalization of a special case of Kogbetliants's theorem.

Theorem E. If the sequence $\{q_n\}$ be non-negative, such that $Q_n \to \infty$, and

$$q_{n+1}/Q_{n+1} = O(q_n/Q_n),$$

then $\Sigma \dfrac{q_n|a_n|}{Q_n} < \infty$, whenever Σa_n is summable $|\bar{N},q_n|$.

In the present chapter, we establish a couple of theorems corresponding to those as stated above, for $|N,p_n q_n|$-method, so as to get Theorems C,D and a result parallel to Theorem E as special cases.

3.4 We prove the following theorems.

Theorem 3.1. Let $\{p_n\}$ and $\{q_n\}$ be such that

(i) $q_{n+1} = O(q_n)$; $\qquad n\,\Delta q_n = O(q_n)$,

(ii) $\sum\limits_{n=0}^{\infty} |c_n| < \infty$,

(iii) $r_n^* = \sum\limits_{v=0}^{n} |p_{n-v}\, q_v| = O(|r_n|)$,

(iv) $\sum_{v=0}^{n} | r_v - r_{v-1} | = O(r_n^*)$,

then the necessary and sufficient condition that $\Sigma a_n c_n$ should be absolutely convergent whenever Σa_n is summable $|N, p_n, q_n|$, is

(3.4.1) $\quad c_n r_n = O(1)$, as $n \to \infty$.

Theorem 3.2. In Theorem 3.1, if the condition (iii) is dropped, then the condition

(3.4.2) $\quad c_n r_n^* = O(1)$, as $n \to \infty$,

is sufficient for the absolute convergence of $\Sigma a_n c_n$.

Remarks. As in DAS [22], it is easily seen that the condition (3.4.2) is stronger than the condition (3.4.1), but these conditions are equivalent when (iii) holds.

3.5 We need the following lemmas for the proof of our theorems.

Lemma 3.1. If

$$t_n^{p*q} = \frac{1}{r_n} \sum_{v=o}^{n} p_{n-v}\, q_v\, s_v\ ,$$

<u>then</u>

$$s_n = \frac{1}{q_n} \sum_{v=o}^{n} c_{n-v}\, r_v\, t_v^{p*q}\ ,$$

<u>where</u> $\{c_n\}$ <u>is</u> <u>defined</u> <u>in</u> (3.2.2).

<u>Proof</u>. By virtue of the identity (3.2.2) we have

$$(3.5.1)\quad \sum_{v=o}^{n} c_{n-v}\, p_v = \begin{cases} 1 & \text{for } n = o\ ; \\ 0 & \text{for } n = 1,2,\ldots \end{cases}$$

Now, by (3.5.1)

$$\sum_{v=o}^{n} c_{n-v}\, r_v\, t_v^{p*q} = \sum_{v=o}^{n} c_{n-v} \sum_{\mu=o}^{v} p_{n-\mu}\, q_\mu\, s_\mu$$

$$= \sum_{\mu=o}^{n} q_\mu\, s_\mu \sum_{v=\mu}^{n} c_{n-v}\, p_{v-\mu} = q_n\, s_n.$$

<u>Lemma</u> 3.2. <u>If</u> $U_n = \sum_{\mu=1}^{\infty} S_{n,\mu}\, a_\mu$ $(n = 1,2,\ldots)$, <u>where</u> $\{S_{n,\mu}\}$ <u>is a double sequence</u>, <u>then a necessary and sufficient condition that the series</u> $\Sigma\ |U_n|$ <u>should be convergent</u>

72817

whenever $\sum |u_n|$ *is convergent, is that* $\sum_n |s_{n,\mu}| \leq K$,
where K *is a constant independent of* μ.

3.6 *Proof of Theorem* 3.1.

Sufficiency : By Lemma 3.1,

$$s_n = \frac{1}{q_n} \sum_{v=0}^{n} c_{n-v}\, r_v\, t_v^{p*q} .$$

Hence, by Abel's transformation, we have

$$a_n = s_n - s_{n-1} = \sum_{v=0}^{n} \left[\frac{c_{n-v}}{q_n} - \frac{c_{n-1-v}}{q_{n-1}} \right] r_v\, t_v^{p*q}$$

$$= \sum_{v=0}^{n-1} \Delta t_v^{p*q} \sum_{\mu=0}^{v} \left[\frac{c_{n-\mu}}{q_n} - \frac{c_{n-1-\mu}}{q_{n-1}} \right] r_\mu$$

$$+ t_n^{p*q} \sum_{\mu=0}^{n} \left[\frac{c_{n-\mu}}{q_n} - \frac{c_{n-1-\mu}}{q_{n-1}} \right] r_\mu .$$

But, since for $n \geq 0$,

$$\text{(3.6.1)} \quad \frac{1}{q_n} \sum_{\mu=0}^{n} c_{n-\mu}\, r_\mu = \frac{1}{q_n} \sum_{\mu=0}^{n} c_{n-\mu} \sum_{\beta=0}^{\mu} p_{\mu-\beta}\, q_\beta$$

$$= \frac{1}{q_n} \sum_{\beta=0}^{n} q_\beta \sum_{\mu=\beta}^{n} c_{n-\mu}\, p_{\mu-\beta} = 1,$$

by virtue of (3.5.1) and Lemma 3.1, we have for $n \geq 1$,

$$(3.6.2)\quad a_n = \sum_{v=0}^{n-1} \Delta t_v^{p*q} \sum_{\mu=0}^{v} \left[\frac{c_{n-\mu}}{q_n} - \frac{c_{n-1-\mu}}{q_{n-1}} \right] r_\mu$$

$$= \sum_{v=0}^{n-1} \Delta t_v^{p*q} \sum_{\mu=0}^{v} \left\{ c_{n-\mu} \left(\frac{r_\mu}{q_n} - \frac{r_{\mu-1}}{q_{n-1}} \right) - \frac{c_{n-v-1}}{q_{n-1}} r_v \right\}$$

$$= \sum_{v=0}^{n-1} \Delta t_v^{p*q} \left\{ \sum_{\mu=0}^{v} c_{n-\mu} \frac{(r_\mu - r_{\mu-1})}{q_n} - \right.$$

$$\left. - \sum_{\mu=0}^{v} c_{n-\mu} r_{\mu-1} \frac{q_{n-1}}{q_n q_{n-1}} - \frac{c_{n-v-1}}{q_{n-1}} r_v \right\}$$

Hence

$$\Sigma \; |\epsilon_n a_n| \leq \sum_{n=1}^{\infty} |\epsilon_n| \left| \sum_{v=0}^{n-1} \Delta t_v^{p*q} \sum_{\mu=0}^{v} c_{n-\mu} \frac{(r_\mu - r_{\mu-1})}{q_n} \right|$$

$$+ \sum_{n=1}^{\infty} |\epsilon_n| \left| \sum_{v=0}^{n-1} \Delta t_v^{p*q} \sum_{\mu=0}^{v} c_{n-\mu} r_{\mu-1} \frac{\Delta q_{n-1}}{q_n q_{n-1}} \right|$$

$$+ \sum_{n=1}^{\infty} |\epsilon_n| \left| \sum_{v=0}^{n-1} \Delta t_v^{p*q} \frac{c_{n-v-1} r_v}{q_{n-1}} \right|$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.}$$

Now, by condition (3.4.1), which becomes equivalent to (3.4.2) when (iii) holds,

$$|C_n| \leq K\, q_n/r_n^* \leq K\, q_n/r_v^* \qquad (n \geq v).$$

Hence

$$\Sigma_1 \leq \sum_{n=1}^{\infty} |C_n| \Big|\sum_{v=0}^{n-1}\Big| \Delta t_v^{p*q} \Big| \sum_{\mu=0}^{v} |c_{n-\mu}| \; \frac{|r_\mu - r_{\mu-1}|}{q_n}$$

$$= \sum_{v=0}^{\infty} \Delta t_v^{p*q} \Big| \sum_{n=v+1}^{\infty} |C_n| \Big| \sum_{\mu=0}^{v} |c_{n-\mu}| \; \frac{|r_\mu - r_{\mu-1}|}{q_n}$$

$$= \sum_{v=0}^{\infty} \Big| \Delta t_v^{p*q} \Big| \sum_{\mu=0}^{v} |r_\mu - r_{\mu-1}| \sum_{n=v+1}^{\infty} |C_n| \; \frac{|c_{n-\mu}|}{q_n}$$

$$\leq K \sum_{v=0}^{\infty} \Big| \Delta t_v^{p*q} \Big| (r_v^*)^{-1} \sum_{\mu=0}^{v} |r_\mu - r_{\mu-1}| \sum_{n=v+1}^{\infty} |c_{n-\mu}|$$

$$\leq K \sum_{v=0}^{\infty} \Big| \Delta t_v^{p*q} \Big| \leq K,$$

by hypotheses.

Next,

$$\Sigma_2 \leq \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | \sum_{n=v+1}^{\infty} |e_n| \sum_{\mu=0}^{v} |c_{n-\mu}| |r_{\mu-1}| \frac{|\Delta q_{n-1}|}{q_n \, q_{n-1}}$$

$$\leq \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | \sum_{\mu=0}^{v} r^*_{\mu-1} \sum_{n=v+1}^{\infty} |e_n| |c_{n-\mu}| \frac{|\Delta q_{n-1}|}{q_n \, q_{n-1}}$$

$$\leq K \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | \frac{1}{r^*_v} \sum_{\mu=0}^{v} r^*_{\mu-1} \sum_{n=v+1}^{\infty} |c_{n-\mu}| \frac{1}{n}$$

$$\leq K \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | \frac{1}{v \, r^*_v} \sum_{\mu=0}^{v} r^*_{\mu-1} \sum_{n=v+1}^{\infty} |c_{n-\mu}|$$

$$\leq K \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | \frac{r^*_{v-1}}{r^*_v}$$

$$\leq K \sum_{v=0}^{\infty} | \Delta t_v^{p*q} |$$

$$< K ,$$

by hypotheses.

Finally,

$$\Sigma_3 \leq \sum_{n=1}^{\infty} |e_n| \sum_{v=0}^{n-1} | \Delta t_v^{p*q} | |r_v| \frac{|c_{n-v-1}|}{|q_{n-1}|}$$

$$\leq \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | |r_v| \sum_{n=v+1}^{\infty} |e_n| \frac{|c_{n-v-1}|}{|q_{n-1}|}$$

$$\leq K \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | r_v^*(r_v^*)^{-1} \sum_{n=v+1}^{\infty} \left(\frac{q_n}{q_{n-1}}\right) |c_{n-v-1}|$$

$$\leq K \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | \sum_{n=v+1}^{\infty} |c_{n-v-1}|$$

$$\leq K \sum_{v=0}^{\infty} | \Delta t_v^{p*q} | \leq K ,$$

by hypotheses.

Thus

$$\Sigma |a_n e_n| \leq \Sigma_1 + \Sigma_2 + \Sigma_3 \leq K ;$$

and this completes the proof of the sufficiency part.

<u>Necessity</u> : By (3.6.2), for $n \geq 1$,

$$e_n a_n = \sum_{v=0}^{n} \Delta t_v^{p*q} \; g_{n,v} ,$$

where

$$(3.6.3) \quad g_{n,v} = \begin{cases} e_n \sum_{\mu=0}^{v} \left(\dfrac{c_{n-\mu}}{q_n} - \dfrac{c_{n-\mu-1}}{q_{n+1}} \right) r_\mu & (v \leq n), \\ 0 & (v > n). \end{cases}$$

Now, by Lemma 3.2 and (3.6.1), a necessary and sufficient condition for $\Sigma\ |e_n\ a_n|$ to be convergent whenever a_n is summable $|N, p_n, q_n|$, i.e. $\sum_n |\ \Delta t_n^{p,q}\ | \leq K$, is that

$$(3.6.4) \quad \sum_{n=v+1}^{\infty} |\ g_{n,v}\ | \ \leq\ K.$$

Hence, it is necessary that $|g_{v+1,v}| = O(1)$, as $v \to \infty$.

But

$$(3.6.5) \quad g_{v+1,v} = e_{v+1} \sum_{\mu=0}^{v} \left[\frac{c_{v+1-\mu}}{q_{v+1}} - \frac{c_{v-\mu}}{q_v} \right] r_\mu$$

$$= e_{v+1} \left\{ \sum_{\mu=0}^{v+1} \frac{c_{v+1-\mu}}{q_{v+1}}\ r_\mu - \sum_{\mu=0}^{v} \frac{c_{v-\mu}}{q_v}\ r_\mu - \frac{c_0\ r_{v+1}}{q_{v+1}} \right\}$$

$$= -\ e_{v+1}\ \frac{c_0\ r_{v+1}}{q_{v+1}} \ .$$

Hence the conditions (3.4.1) is necessary.

<u>Remarks</u>. As remarked and proved in Das [22] both the

conditions (3.6.5) and (3.5.4) are equivalent.

3.7 *Proof of Theorem* 3.2. This is evident from the proof of the sufficiency part of Theorem 3.1.

3.8 The following corollaries are immediate :

Corollary 3.1. *Let* $\{q_n\}$ *be such that* $q_n > 0$, $q_{n+1} = O(q_n)$ *and* $n \Delta q_n = O(q_n)$. *Then the necessary and sufficient condition that* $\Sigma |\epsilon_n a_n| < \infty$, *whenever* Σa_n *is summable* $|\bar{N}, q_n|$, *is that*

$$\epsilon_n = O\left(\frac{q_n}{Q_n}\right).$$

This yields a result corresponding to Theorem E of Peyerimhoff.

Corollary 3.2. *Let* $p \in \mathcal{M}$, *and* $\{q_n\}$ *be such that* $q_n > 0$, $q_{n+1} = O(q_n)$ *and* $n \Delta q_n = O(q_n)$. *Then the necessary and sufficient condition that* $\Sigma \epsilon_n a_n$ *should be absolutely convergent whenever* Σa_n *is summable* $|N, p_n, q_n|$ *is*

$$\epsilon_n r_n = O(q_n), \text{ as } n \to \infty.$$

For obtaining this result we appeal to a result of Kaluza (see [40], Theorem 22) which states that when $\{p_n\} \in \mathcal{M}$, $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent for $|x| \leq 1$.

Chapter IV

$|N,p_n,q_n|$-SUMMABILITY FACTORS OF INFINITE SERIES

4.1 <u>Definitions and Notations</u> : In addition to the definitions and notations of Chapter III, we use throughout the following notations :

$$\widetilde{t}_n^{\,q} = (Q_n)^{-1} \sum_{v=1}^{n} q_{v-1}\, a_v \ ;$$

$$r_n^{\mu} = \sum_{k=\mu}^{n} p_{n-k}\, q_k \quad (\mu \leq n) \ ;$$

$$r_n^{*} = \sum_{k=0}^{n} \Delta p_{n-k}\, q_k \ ;$$

$$r_n^{*\mu} = \sum_{k=\mu}^{n} \Delta p_{n-k}\, q_k \quad (\mu \leq n) \ ;$$

$$R(n,\mu) = (r_n\, r_n^{*\mu} - r_n^{\mu}\, r_n^{*}) \ ;$$

$$W_n = \sum_{\mu=1}^{n} \frac{\Delta_\mu R(n,\mu)}{r_n\, r_{n-1}}\, \epsilon_{\mu+1}\, \widetilde{t}_\mu^{\,q} \, .$$

4.2 <u>Introduction</u> : If a given infinite series is not summable $|N,p_n,q_n|$, but the n^{th} total variation of the

(N,p_n,q_n)-mean is of certain order, say μ_n, i.e.,

$$\sum_{k=1}^{n} | t_n^{p*q} - t_{n-1}^{p*q} | = O(\mu_n),$$

where $\{\mu_n\}$ is a positive non-decreasing sequence then the very natural question arises that how to choose a sequence $\{\epsilon_n\}$ so that the series $\Sigma\, \epsilon_n a_n$ may be summable $|N,p_n,q_n|$?

In this direction several researchers obtained different result for various summability methods (See e.g., SUNOUCHI [74], G.D.DIKSHIT [27], AHMAD [8], etc). AHMAD [8] generalised a number of known results ([46], [54], [74], [87], [92a]) for $|N,p_n|$-summability and established the following.

<u>Theorem</u> A. Let $p_0 > 0$, $p_n > 0 (n = 1,2,\ldots)$, and let $\{p_n\}$ be non-increasing. If

$$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\mu_n),$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, and if the sequence $\{\epsilon_n\}$ is such that

(i) $\epsilon_n \mu_n = O(1)$, $n \Delta \mu_n = O(\mu_n)$

(ii) $\Sigma\, n\, \mu_n \mid \Delta^2 \epsilon_n \mid < \infty$,

then the series $\Sigma(n+1)^{-1}\, P_n\, \epsilon_n\, a_n$ is summable $|N, p_n|$.

KHAN [44] has recently proved the following theorem for $|\bar{N}, q_n|$-method.

<u>Theorem</u> B. Let $q_n > 0 (n = 0,1,2,\ldots)$ and $(n+1)q_n \leq K\, Q_n$. If

$$\sum_{v=1}^{n} (q_v/Q_{v-1}) \mid \widetilde{t}_v^{\,q} \mid = O(\mu_n),$$

where $\{\mu_n\}$ is a positive monotonic non-decreasing sequence, and if the sequences $\{\epsilon_n\}$ and μ_n are such that

(i) $\epsilon_n \mu_n = O(1)$, $\quad \mu_n = O\left(\dfrac{| Q_n |}{q_n} \mu_n \right)$,

(ii) $\displaystyle\sum_{n=1}^{\infty} Q_n \left| \Delta \left(\frac{1}{q_n}\right) \right| \mu_n \mid \Delta \epsilon_{n+1} \mid < \infty$,

(iii) $\displaystyle\sum_{n=1}^{\infty} (Q_n/q_n)\, \mu_n \mid \Delta^2 \epsilon_n \mid < \infty$, then the series $\Sigma\, \epsilon_n a_n$ is summable $|\bar{N}, q_n|$.

Our object in this chapter is to prove a couple of analogous results for $|N, p_n, q_n|$-method, the second of which includes Theorems A and B as special cases.

4.3 We establish the following theorems ;

<u>Theorem</u> 4.1. <u>Let</u> $\{p_n\}$ <u>and</u> $\{q_n\}$ <u>be positive sequences</u> <u>such that</u> $(n+1)q_n \leq K\, Q_n$, $n \Delta q_n = O(q_n)$*, <u>and</u>

$$\text{(i)} \qquad \sum_{n=\mu}^{\infty} \frac{p(n,\mu)}{r_n\, r_{n-1}} \leq K .$$

<u>If</u>

$$\sum_{v=1}^{n} \frac{q_v}{Q_{v-1}} \; | \widetilde{t}_v^{\,q} | = O(\lambda_n),$$

<u>where</u> $\{\lambda_n\}$ <u>is a positive monotonic non-decreasing sequence</u>, <u>and if the sequences</u> $\{\epsilon_n\}$ <u>and</u> $\{\lambda_n\}$ <u>are such that</u>

$$\text{(i)} \qquad \lambda_n \epsilon_n = O(1); \; \Delta \lambda_n = O\left(\frac{|\Delta q_n|}{q_n} \lambda_n \right),$$

---

* This condition is void when $p_n = 1$, for all $n \geq 0$.

(ii) $\sum_{n=1}^{\infty} Q_n \mid \Delta(\frac{1}{q_n}) \mid \lambda_n \mid \Delta e_{n+1} \mid < \infty$,

(iii) $\sum_{n=1}^{\infty} \frac{Q_n}{q_n} \lambda_n \mid \Delta^2 e_n \mid < \infty$,

then the necessary and sufficient condition for $|N,p_n,q_n|$-*summability of the series* $\Sigma e_n a_n$ *is that*

$$\sum_{n=1}^{\infty} | W_n | \leq K .$$

Theorem 4.2. Let $\{p_n\}$ *and* $\{q_n\}$ *be positive sequences such that* $(n+1)q_n \leq K Q_n$, $n \Delta q_n = O(q_n)^*$, *and*

(i) $r_n^* = \sum_{v=0}^{n} \Delta_v p_{n-v} q_v = O(q_n)$,

(ii) $\frac{r_{n-1}}{r_n} = O(1)$,

(iii) $\sum_{n=\mu}^{\infty} \frac{|R(n,\mu)|}{r_n r_{n-1}} \leq K$,

(iv) $\sum_{n=\mu}^{\infty} \frac{| \Delta \{R(n,\mu)\} |}{r_n r_{n-1}} = O(\frac{q_\mu}{r_{\mu-1}})$.

---

* This condition is void when $p_n = 1$, for all $n \geq 0$.

<u>If</u>

$$\sum_{\mu=1}^{n} \frac{q_\mu}{Q_{\mu-1}} \, | \, \tilde{t}_\mu^{\,q} \, | = O(\lambda_n),$$

<u>where $\{\lambda_n\}$ is a positive monotonic non-decreasing sequence, and if the sequences $\{c_n^*\}$ and $\{\lambda_n\}$ are such that</u>

$$\text{(i)} \qquad \lambda_n \, c_n^* = O(1) \; ; \Delta \lambda_n = O\left(\frac{|\Delta q_n|}{q_n} \, \lambda_n \right),$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} Q_n \, | \, \Delta \left(\frac{1}{q_n}\right) | \, \lambda_n \, | \, \Delta \, c_{n+1}^* \, | < \infty ,$$

$$\text{(iii)} \quad \sum_{n=1}^{\infty} \frac{Q_n}{q_n} \, \lambda_n | \, \Delta^2 \, c_n^* \, | < \infty ,$$

<u>then</u> $\sum \frac{r_n}{R_n} \, c_n^* \, a_n$ <u>is summable</u> $|N, p_n, q_n|$.

It is to be noted that if (i) $q_n = 1$ for all n and (ii) $p_n = 1$ for all n then our Theorem 4.2 includes Theorem A and Theorem B respectively.

4.4 We need the following lemmas for the proof of our theorems.

<u>Lemma</u> 4.1 ([44], Lemma 1). <u>Let</u> $q_n > 0$, <u>for all</u> $n \geq 0$,

such that $(n+1)q_n \leq K\, Q_n$. *If* $c_n\lambda_n = O(1)$ *and*

$$\sum_{n=1}^{\infty} (Q_n/q_n)\, \lambda_n \mid \Delta^2 c_n \mid < \infty ,$$

where $\{\lambda_n\}$ *is a positive monotonic non-decreasing sequence, then*

$$\sum_{n=1}^{\infty} \lambda_n \mid \Delta c_n \mid < \infty .$$

Lemma 4.2 ([14], Lemma 2). *If the sequences* $\{\lambda_n\}$ *and* $\{c_n\}$ *satisfy the same conditions as in Theorem* 4.1, *then*

$$(Q_{n-1}/q_n)\, \lambda_n \mid \Delta c_n \mid = O(1), \text{ as } n \to \infty.$$

Lemma 4.3. *Under the hypotheses of Theorem* 4.1 *with the* c_n *replaced by* α_n, *we have*

(a) $$\sum_{\mu=1}^{\infty} \mid \Delta \alpha_\mu \mid \mid \widetilde{t}_\mu^{\,q} \mid \leq K ;$$

(b) $$\sum_{\mu=1}^{\infty} \frac{q_\mu}{Q_{\mu-1}} \mid \Delta \alpha_\mu \mid \mid \widetilde{t}_\mu^{\,q} \mid \leq K ;$$

(c) $$\sum_{\mu=1}^{\infty} \frac{q_\mu}{Q_{\mu-1}} \mid \alpha_{\mu+1} \mid \mid \widetilde{t}_\mu^{\,q} \mid \leq K .$$

Proof. (a) We have, as $m \to \infty$,

$$\sum_{\mu=1}^{m} |\Delta a_\mu| \, |\tilde{t}_\mu^{\,q}| = \sum_{\mu=1}^{m} \frac{q_{\mu-1}}{q_\mu} \, |\Delta a_\mu| \, \frac{q_\mu}{q_{\mu-1}} \, |\tilde{t}_\mu^{\,q}|$$

$$= O\Big(\sum_{\mu=1}^{m-1} \frac{q_{\mu-1}}{q_\mu} \, |\Delta^2 a_\mu| \lambda_\mu\Big) + O\Big(\sum_{\mu=1}^{m-1} |\Delta a_{\mu+1}| \lambda_\mu\Big)$$

$$+ O\Big(\sum_{\mu=1}^{m-1} q_\mu \Big|\Delta\Big(\frac{1}{q_\mu}\Big)\Big| \, |\Delta a_{\mu+1}| \, \lambda_\mu\Big)$$

$$+ O\Big(\frac{q_{m-1}}{q_m} \, |\Delta a_m| \lambda_m\Big)$$

$$= O(1),$$

by hypotheses and Lemmas 4.1 and 4.2.

(b) We see that, as $m \to \infty$,

$$\sum_{\mu=1}^{m} \frac{q_\mu}{q_{\mu-1}} \, |\Delta a_\mu| \, |\tilde{t}_\mu^{\,q}| = O\Big(\sum_{\mu=1}^{m-1} |\Delta^2 a_\mu| \, \lambda_\mu\Big) +$$

$$+ O(|\Delta a_m| \lambda_m)$$

$$= O(1),$$

by hypotheses and Lemma 4.1.

(c) We see that, as $m \to \infty$,

$$\sum_{\mu=1}^{m} \frac{q_\mu}{Q_{\mu-1}} |\alpha_{\mu+1}| \, |\tilde{t}_\mu^{\,q}| = O\Big(\sum_{\mu=1}^{m-1} |\Delta \alpha_{\mu+1}| \lambda_\mu\Big) + O(|\alpha_{m+1}| \lambda_m)$$

$$= O(1),$$

by hypotheses and Lemma 4.1.

This completes the proof of the lemma.

4.5 <u>Proof of Theorem</u> 4.1. Let t_n^{p*q} denote the n^{th} (N, p_n, q_n)-mean of the series $\sum \varepsilon_n a_n$. Then, by definition

$$t_n^{p*q} - t_{n-1}^{p*q} = \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \varepsilon_\mu a_\mu$$

$$= \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \frac{\varepsilon_\mu}{Q_{\mu-1}} Q_{\mu-1} a_\mu$$

$$= \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} \Delta \Big[(r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \frac{\varepsilon_\mu}{Q_{\mu-1}} \Big] Q_\mu \tilde{t}_\mu^{\,q}$$

$$+ \frac{1}{r_n r_{n-1}} (r_n r_n^{*n+1} - r_n^{n+1} r_n^{*}) \frac{\varepsilon_{n+1}}{Q_n} Q_n \tilde{t}_n^{\,q}$$

$$= \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} \Delta \Big[(r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \frac{\varepsilon_\mu}{Q_{\mu-1}} \Big] Q_\mu \tilde{t}_\mu^{\,q}$$

(since $r_n^{*(n+1)} = \sum_{k=n+1}^{n} p_{n-1} q_k = 0$, $r_n^{(n+1)} = \sum_{k=n+1}^{n} p_{n-k} q_k = 0$)

By applying Abel's transformation, we have

$$t_n^{p*q} - t_{n-1}^{p*q} = \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \frac{\Delta e_\mu}{q_{\mu-1}} q_\mu \tilde{t}_\mu^{\,q}$$

$$+ \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \frac{e_{\mu+1} q_\mu}{q_\mu q_{\mu-1}} q_\mu \tilde{t}_\mu^{\,q}$$

$$+ \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n \Delta r_n^{*\mu} - \Delta r_n^{\mu} r_n^{*}) \frac{e_{\mu+1}}{q_\mu} q_\mu \tilde{t}_\mu^{\,q}$$

$$= \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \Delta e_\mu \tilde{t}_\mu^{\,q} \left(1 + \frac{q_\mu}{q_{\mu-1}}\right)$$

$$+ \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) e_{\mu+1} \frac{q_\mu}{q_{\mu-1}} \tilde{t}_\mu^{\,q}$$

$$+ \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n \Delta_\mu r_n^{*\mu} - \Delta_\mu r_n^{\mu} r_n^{*}) e_{\mu+1} \tilde{t}_\mu^{\,q}$$

$$= \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \Delta e_\mu \tilde{t}_\mu^{\,q}$$

$$+ \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) \Delta e_\mu \frac{q_\mu}{q_{\mu-1}} \tilde{t}_\mu^{\,q}$$

$$+ \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n r_n^{*\mu} - r_n^{\mu} r_n^{*}) e_{\mu+1} \frac{q_\mu}{q_{\mu-1}} \tilde{t}_\mu^{q}$$

$$+ \frac{1}{r_n r_{n-1}} \sum_{\mu=1}^{n} (r_n q_\mu \Delta_\mu p_{n-\mu} - p_{n-\mu} q_\mu r_n^{*}) e_{\mu+1} \tilde{t}_\mu^{q} ,$$

so that

$$t_n^{p*q} - t_{n-1}^{p*q} = \sum_{\mu=1}^{n} \frac{R(n,\mu)}{r_n r_{n-1}} \Delta e_\mu \tilde{t}_\mu^{q} + \sum_{\mu=1}^{n} \frac{R(n,\mu)}{r_n r_{n-1}} \Delta e_\mu \frac{q_\mu}{q_{\mu-1}} \tilde{t}_\mu^{q}$$

$$+ \sum_{\mu=1}^{n} \frac{R(n,\mu)}{r_n r_{n-1}} e_{\mu+1} \frac{q_\mu}{q_{\mu+1}} \tilde{t}_\mu^{q}$$

$$+ \sum_{\mu=1}^{n} \frac{\Delta_\mu \{R(n,\mu)\}}{r_n r_{n-1}} e_{\mu+1} \tilde{t}_\mu^{q}$$

$$(4.5.1) \qquad = L_{n,1} + L_{n,2} + L_{n,3} + V_n , \text{ say.}$$

We see that,

$$\sum_{n=1}^{\infty} |L_{n,1}| \le \sum_{n=1}^{\infty} \sum_{\mu=1}^{n} \frac{|R(n,\mu)|}{r_n r_{n-1}} |\Delta e_\mu| |\tilde{t}_\mu^{q}|$$

$$\le \sum_{\mu=1}^{\infty} |\Delta e_\mu| |\tilde{t}_\mu^{q}| \sum_{n=\mu}^{\infty} \frac{|R(n,\mu)|}{r_n r_{n-1}}$$

$$\leq K \sum_{\mu=1}^{\infty} | \Delta e_{\mu} | \, | \widetilde{t}_{\mu}^{\,q} | \quad \text{(by hypothesis (i))}$$

$$\leq K ,$$

by the application of Lemma 4.3 (a).

Next, we have

$$\sum_{n=1}^{\infty} |L_{n,2}| \leq \sum_{n=1}^{\infty} \sum_{\mu=1}^{n} \frac{q_{\mu}}{Q_{\mu-1}} \frac{|R(n,\mu)|}{r_n \, r_{n-1}} \, | \Delta e_{\mu} | \, | \widetilde{t}_{\mu}^{\,q} |$$

$$= \sum_{\mu=1}^{\infty} \frac{q_{\mu}}{Q_{\mu-1}} \, | \Delta e_{\mu} | \, | \widetilde{t}_{\mu}^{\,q} | \sum_{n=\mu}^{\infty} \frac{|R(n,\mu)|}{r_n \, r_{n-1}}$$

$$\leq K \sum_{\mu=1}^{\infty} \frac{q_{\mu}}{Q_{\mu-1}} \, | \Delta e_{\mu} | \, | \widetilde{t}_{\mu}^{\,q} | \quad \text{(by hypothesis (i))},$$

$$\leq K ,$$

by the use of Lemma 4.3(b).

Next, we have

$$\sum_{n=1}^{\infty} |L_{n,3}| \leq \sum_{n=1}^{\infty} \sum_{\mu=1}^{n} \frac{q_{\mu}}{Q_{\mu-1}} \frac{|R(n,\mu)|}{r_n \, r_{n-1}} \, |e_{\mu+1}| \, | \widetilde{t}_{\mu}^{\,q} |$$

$$= \sum_{\mu=1}^{\infty} |\epsilon_{\mu+1}| \frac{q_\mu}{Q_{\mu-1}} |\widetilde{t}_\mu^{\,q}| \sum_{n=\mu}^{\infty} \frac{|R(n,\mu)|}{r_n r_{n-1}}$$

$$= K \sum_{\mu=1}^{\infty} |\epsilon_{\mu+1}| \frac{q_\mu}{Q_{\mu-1}} |\widetilde{t}_\mu^{\,q}| \quad \text{(by hypothesis (i)),}$$

$$\leq K$$

by appealing Lemma 4.3 (c).

Thus

$$\sum_{n=1}^{\infty} |L_{n,r}| < \infty, \quad \text{for } r = 1,2,3.$$

Now, since we can write

$$(4.5.2) \qquad W_n = (t_n^{p*q} - t_{n-1}^{p*q}) - L_{n,1} - L_{n,2} - L_{n,3},$$

therefore, in order that $\Sigma \mid t_n^{p*q} - t_{n-1}^{p*q} \mid \leq K$, whenever $\sum_{n=1}^{\infty} |L_{n,r}| \leq K$, ($r = 1,2,3$), by (4.5.1) and (4.5.2), it is necessary and sufficient that

$$\sum_{n=1}^{\infty} | W_n | \leq K .$$

This terminates the proof of Theorem 4.1.

4.6 __Proof of Theorem__ 4.2. We give the proof of Theorem 4.2, by taking $e_n = \frac{r_n}{Q_n} e_n^*$ in Theorem 4.1.

Therefore, in order that $\sum_{n=1}^{\infty} | t_n^{p*q} - t_{n-1}^{p*q} | \leq K$, by virtue of (4.1.2), it is sufficient to show that

$$\sum_{n=1}^{\infty} |L_{n,r}| \leq K \quad (r = 1,2,3),$$

and $$\sum_{n=1}^{\infty} | W_n | \leq K .$$

Since,

$$\Delta e_n = \frac{r_n}{Q_n} \Delta e_n^* + \frac{r_n^*}{Q_n} e_{n+1}^* + \frac{q_{n+1}}{Q_n Q_{n+1}} r_n e_{n+1}^* ,$$

we see that, as in the proof of Theorem 4.1.

$$\sum_{n=1}^{\infty} |L_{n,1}| \leq K \sum_{\mu=1}^{\infty} |\Delta e_\mu| \; |\widetilde{t}_\mu^{\,q}|$$

$$\leq K \sum_{\mu=1}^{\infty} |\Delta e_\mu^*| |\widetilde{t}_\mu^{\,q}| + K \sum_{\mu=1}^{\infty} \frac{q_\mu}{Q_{\mu-1}} |e_{\mu+1}^*| |\widetilde{t}_\mu^{\,q}|$$

$$\leq K ,$$

by hypotheses and Lemma 4.3(a) and (c).

Next, we have

$$\sum_{n=1}^{\infty} |L_{n,2}| \leq K \sum_{\mu=1}^{\infty} |\Delta e_\mu| \frac{q_\mu}{Q_{\mu-1}} |\tilde{t}_\mu^{\,q}|$$

$$\leq K \sum_{\mu=1}^{\infty} |\Delta e^*_\mu| \frac{q_\mu}{Q_{\mu-1}} |\tilde{t}_\mu^{\,q}| + K \sum_{\mu=1}^{\infty} |e^*_{\mu+1}| \frac{q_\mu}{Q_{\mu-1}} |\tilde{t}_\mu^{\,q}|$$

$$\leq K,$$

by hypotheses and Lemmas 4.3(b) and (c).

Again, we have

$$\sum_{n=1}^{\infty} |L_{n,3}| \leq K \sum_{\mu=1}^{\infty} | e_{n+1} | \frac{q_\mu}{Q_{\mu-1}} | \tilde{t}_\mu^{\,q}|$$

$$\leq K \sum_{\mu=1}^{\infty} | e^*_{\mu+1}| \frac{q_\mu}{Q_{\mu-1}} |\tilde{t}_\mu^{\,q}|$$

$$\leq K,$$

by hypotheses and Lemma 4.3(c).

Finally, we have

$$\sum_{n=1}^{\infty} |V_n| \leq \sum_{n=1}^{\infty} \sum_{\mu=1}^{n} \frac{|\Delta_\mu \{R(n,\mu)\}|}{r_n r_{n-1}} |\varepsilon_{\mu+1}| |\widetilde{t}_\mu^{\,q}|$$

$$= \sum_{\mu=1}^{\infty} |\varepsilon_{\mu+1}| |\widetilde{t}_\mu^{\,q}| \sum_{n=\mu}^{\infty} \frac{|\Delta_\mu \{R(n,\mu)\}|}{r_n r_{n-1}}$$

$$\leq K \sum_{\mu=1}^{\infty} |\varepsilon_{\mu+1}| \frac{q_\mu}{r_{\mu-1}} |\widetilde{t}_\mu^{\,q}| \text{ (by hypothesis (iv)),}$$

$$\leq K \sum_{\mu=1}^{\infty} \frac{Q_{\mu-1}}{r_{\mu-1}} |\varepsilon_{\mu+1}| \frac{q_\mu}{Q_{\mu-1}} |\widetilde{t}_\mu^{\,q}|$$

$$\leq K \sum_{\mu=1}^{\infty} \frac{Q_{\mu-1}}{Q_\mu} \cdot \frac{r_\mu}{r_{\mu-1}} |\varepsilon^*_{\mu+1}| \frac{q_\mu}{Q_{\mu-1}} |\widetilde{t}_\mu^{\,q}|$$

$$\leq K \sum_{\mu=1}^{\infty} |\varepsilon^*_{\mu+1}| \frac{q_\mu}{Q_{\mu-1}} |\widetilde{t}_\mu^{\,q}|$$

$$\leq K,$$

by hypotheses and Lemma 4.3(c).

This completes the proof of Theorem 4.2.

Chapter V

ABSOLUTE MATRIX SUMMABILITY FACTORS OF A FOURIER SERIES

5.1. Definitions and Notations. Let Σa_n be an infinite series with sequence of partial sums $\{s_n\}$. Let $T = (a_{nk})$ be an infinite matrix with real, or complex elements. Then the T-transform of $\{s_n\}$, given by the matrix multiplication :

(5.1.1) $$t_n = \sum_{k=0}^{\infty} a_{nk}\, s_k ,$$

(assuming that 't_n' exists, for every $n = 0,1,2,\ldots$), defines the matrix transform of the sequence $\{s_n\}$, or the series Σa_n, generated by the elements of the matrix T. If

$$\lim_{n \to \infty} t_n = s ,$$

the series Σa_n, or the sequence $\{s_n\}$, is said to be summable by the matrix method (T), or simply (T)-summable, to s (s being finite). The series Σa_n, or the sequence $\{s_n\}$, is said to be absolutely summable (T), or simply summable $|T|$, if $\{t_n\} \in BV$.

As given in Chapter I, the necessary and sufficient conditions for the (T)-method to be regular, are that ;

(i) there is a constant K, such that

$$\sum_{k=0}^{\infty} |a_{n,k}| < K \text{ , for every } n,$$

(ii) for every k,

$$\lim_{n\to\infty} a_{nk} = 0,$$

and

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1.$

If matrix elements $a_{n,k} = 0$, for every $k > n$, then the matrix T is called the <u>triangular matrix</u>.

In particular if,

$$a_{n,k} = \frac{p_{n-k}}{P_n} \quad (k \leq n),$$

$$= 0 \quad (k > n),$$

where $P_n = p_0 + p_1 + \dots + p_n \neq 0$, then t_n, defined by

(5.1.1), is the same as Nörlund mean ([68], [106]), or (N,p_n)-mean, generated by the sequence of coefficients $\{p_n\}$, real or complex, and the corresponding absolute summability method is absolute Nörlund summability method or $|N,p_n|$-method. $|N,(n+1)^{-1}|$-method (for $p_n = (n+1)^{-1}, P_n \sim \log n$) is called the absolute harmonic summability method.

Similarly, if

$$a_{n,k} = \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}}, \ \alpha > -1, \text{ for } k \leq n,$$

$$= 0, \text{ for } k > n,$$

then t_n is the same as (C,α)-mean (See [40]), or the familiar Cesàro mean of order $\alpha(\alpha > -1)$; and $|T|$ is the same as $|C,\alpha|$.

5.2. Let $f(t)$ be a periodic function with period 2π, and integrable (L), i.e. integrable in the sense of Lebesgue, over $(-\pi,\pi)$. Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is,

$$(5.2.1) \qquad \int_{-\pi}^{\pi} f(t)\, dt = 0.$$

Then

$$(5.2.2) \qquad f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t)$$

denotes the Fourier series of $f(t)$.

We write

$$\phi(t) = \phi_x(t) = \frac{1}{2}\left\{f(x+t) + f(x-t)\right\},$$

$$\lambda(n) = \lambda_n \text{ , and } \Delta\lambda_n = \lambda_n - \lambda_{n+1} \text{ .}$$

5.3. Introduction. Concerning the $|C,\alpha|$-summability of Fourier series, in 1936, BOSANQUET [16] proved the following theorem.

Theorem A. If $\phi(t) \in BV(0,\pi)$, then the series $\sum_{n=1}^{\infty} A_n(t)$, at $t = x$, is summable $|C,\alpha|$, for every $\alpha > 0$.

PATI [72] for the first time in 1959 obtained the generalisation of this result for $|N,p_n|$-summability. Subsequently a number of authors extensively discussed variants of Pati's result (See Chapter I, Section 1.7(b)). In 1964, T. SINGH [92] proved the following theorem.

Theorem B. If $\phi(t) \in BV(0,\pi)$ and $\{p_n\}$ be a non-negative and non-increasing sequence such that $\{\Delta p_n\}$ is non-increasing, and

$$\sum_{v=1}^{n} \frac{P_v}{v+1} \leq K \; P_n \, ,$$

then the series $\sum_{n=1}^{\infty} A_n(t)$, at $t = x$, is summable $|N,p_n|$.

On the other hand, since $\phi(t) \in BV(0,\pi)$ is not sufficient to ensure $|N,(n+1)^{-1}|$-summability of the Fourier series of $f(t)$, at $t = x$, (See [75], p.17), O.P.VARSHNEY [102] proved the following.

Theorem C. If $\phi(t) \in BV(0,\pi)$, then the series $\sum_{n=1}^{\infty} \frac{A_n(t)}{\log(n+1)}$, at $t = x$, is summable $|N,(n+1)^{-1}|$.

Generalising Theorem A in another direction, in 1967, SINGH [92a] proved the following theorem for $|N,p_n|$-summability factors of Fourier series, so as to include Theorem C also as a special case when $p_n = (n+1)^{-1}$.

Theorem D. If $\phi(t) \in BV(0,\pi)$, then the series $\Sigma \frac{(n+1)p_n}{P_n} A_n(t)$, at $t = x$, is summable $|N,p_n|$, where the sequence $\{p_n\}$ is real, non-negative and non-increasing such that

(i) $\{(n+1)p_n/ P_n\} \in BV$,

and

(ii) the sequence $\{\Delta p_n\}$ is non-increasing.

In 1969, KANNO [43] generalised this theorem (Theorem D) avoiding the condition (i) and proved :

Theorem E. Let $\{p_n\}$ and $\{\Delta p_n\}$ be both real, non-negative and non-increasing sequence, and let $\lambda(t)$, $t > 0$, be a positive, non-decreasing function such that $\{\lambda_n/P_n\}$ is non-increasing.

If the conditions

$$\sum_{n=k}^{\infty} \frac{\lambda_n p_n}{P_n^2} = O(\frac{\lambda_k}{P_k})^{*},$$

* When $\lambda(t)$ is a constant function this condition is satisfied automatically.

and

$$\int_0^\pi \lambda(C/t) \ |d\phi(t)| < \infty ,$$

for some constant $C > 0$, hold, then the series

$$\sum_{n=0}^{\infty} \frac{(n+1)p_n}{P_n} \ \lambda_n \ A_{n+1}(t),$$

at $t = x$, is summable $|N, p_n|$.

Later on, in 1971, KISHORE and HOTA [40] generalized Theorems A and B, by extending them to absolute matrix summability. They proved the following :

<u>Theorem</u> F. Let $T = (a_{nk})$ be an infinite triangular matrix, and let us write $\sum_{v=k}^{n} a_{n,v} = A_{n,k}$ and assume $A_{n,o} = 1$ for every $n \geq o$. If $\phi(t) \in BV(o,\pi)$, then the series $\sum_{n=1}^{\infty} A_n(t)$, at $t = x$, is summable $|T|$, provided that

(i) $\left\{a_{n,k}\right\}_{k=o}^{n}$ and $\left\{a_{n-1,k} - a_{n,k+1}\right\}_{k=o}^{n-1}$ are non-negative and non-decreasing sequences with respect to k.

(ii) $\left\{a_{n,k+1} - a_{n,k}\right\}_{k=o}^{n-1}$ is a non-decreasing sequence with respect to k, and that

(iii) $\sum_{n=k+1}^{\infty} \frac{A_{n,n-k}}{n} < K$, for every k.

The object of this chapter is to generalise Theorems D and E, by extending them to absolute matrix summability, in the manner of Kishore and Hota's theorem (Theorem F).

5.4. We prove the following theorem.

Theorem 5.1. Let $T = (a_{nk})$ be a regular infinite triangular matrix, and let us write $\sum_{v=k}^{n} a_{n,v} = A_{n,k}$, and assume that $A_{n,o} = 1$ for every $n \geq 0$, and that

(i) $a_{2n,2n-k} = O(a_{n,n-k})$; $n \Delta a_{n,o} = O(a_{n,o})$;

(ii) $\left\{a_{n,k}\right\}_{k}^{n}$ is a positive sequence such that $\left\{(a_{n,k}-a_{n,o})\, a_{k,o}\right\}_{k=o}^{n}$ is non-decreasing sequence with respect to k ;

(iii) $\left\{a_{n-1,k} - a_{n,k+1}\right\}_{k=o}^{n-1}$ is a non-negative and non-decreasing sequence with respect to k ;

(iv) $\left\{a_{n,k+1} - a_{n,k}\right\}_{k=o}^{n-1}$ is a non-negative and non-decreasing sequence with respect to k.

Further, let $\lambda(t)$, $t > 0$, *be a positive, non-decreasing* function such that

(a) $n \Delta \lambda_n = O(\lambda_n)$,

(b) $\{\lambda_n a_{n,o}\}$ *is a non-increasing sequence*, and

(c) *for every* $k \geq 0$,

$$\sum_{n=k}^{\infty} A_{n,n-k} \lambda_n a_{n,o} = O(\lambda_k).$$

If

$$(5.4.1) \qquad \int_0^{\pi} \lambda(C/t) \quad |d\phi(t)| < \infty ,$$

for some constant $C > 0$, *holds, then the series*

$$\sum_{n=1}^{\infty} \lambda_n \left\{(n+1)\, a_{n,o}\right\} A_{n+1}(t) ,$$

at $t = x$, *is summable* $|T|$.

Remark. It is to be noted that the conditions (ii) and (b) together imply that $\left\{a_{k,o}\right\}$ is a non-increasing sequence and $\left\{a_{n,k}\right\}_{k=0}^{n}$ is a positive non-decreasing sequence with respect to k.

3.3. We require the following lemmas for the proof of our theorems.

Lemma 1 ([48], Lemma 2). If $\left\{a_{n,k}\right\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k, then for $0 \le a < b < \infty$, $0 \le t \le \pi$, and for every n

$$\left| \sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)t} \right| \le K A_{n,n-\tau'},$$

Where $\tau' = [1/t]$, $0 < t < \pi$.

Lemma 2 ([48], Lemma 3). If $\left\{a_{n,k}\right\}_{k=0}^{n}$ is non-negative and non-decreasing sequence with respect to k such that $A_{n,0} = 1$, then as $n \to \infty$, $a_{n,k} = O\left(\frac{1}{n-k+1}\right)$, uniformly for all $k \le n$.

From this it follows that $a_{n,0} = O(1/n)$.

Lemma 3 ([48], Lemma 4). If $\left\{a_{n,k}\right\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k such that $A_{n,0}=1$, then $(A_{n-1,k} - A_{n,k+1})$ is non-negative sequence with respect to k and is equal to $O(1/n)$, as $n \to \infty$, uniformly for all $k \le n$.

Lemma 4 ([48], *Lemma* 5). *If* $\left\{a_{n,k}\right\}_{k=0}^{n}$ *is a non-negative and non-decreasing sequence with respect to* k *such that* $A_{n,0} = 1$ *and* $\left\{a_{n-1,k} - a_{n,k+1}\right\}_{k=0}^{n-1}$ *is non-negative and non-decreasing with respect to* k, *then*

$$\left\{ \frac{A_{n,n-k+1} - A_{n-1,n-k} + a_{n,0}}{n-k} \right\}$$

is a non-negative and non-decreasing sequence with respect to k *for* $k < n$.

Lemma 5 ([48], *Lemma* 6). *If* $\left\{a_{n,k}\right\}_{k=0}^{n}$ *is a non-negative and non-decreasing sequence with respect to* k, *such that* $\left\{a_{n,k+1} - a_{n,k}\right\}_{k=0}^{n-1}$ *is non-decreasing, then*

$$\left\{ \frac{a_{n,n-k} - a_{n,0}}{n-k} \right\}$$

is non-negative and non-increasing with respect to k *for* $k < n$.

Lemma 6 ([48], *Lemma* 7). *If* $\left\{a_{n,k}\right\}_{k=0}^{n}$ *and* $\left\{a_{n-1,k} - a_{n,k+1}\right\}_{k=0}^{n-1}$ *are both non-negative and non-decreasing with respect to* k, *then*

$$\left\{ \frac{A_{n-1,k} - A_{n,k+1}}{k} \right\}_{k=1}^{n}$$

<u>is a non-negative and non-increasing sequence with respect to</u> k.

<u>Lemma</u> 7 ([48], <u>Lemma</u> 1). <u>If</u> $\left\{ a_{n,k} \right\}_{k=0}^{n}$ <u>is non-negative and non-decreasing sequence with respect to</u> k, <u>then</u>

$$t^{-1} a_{n,n-\tau'} \leq A_{n,n-\tau'} ,$$

where $\tau' = [1/t]$, $0 < t < \pi$.

<u>Lemma</u> 8. <u>If for a regular infinite triangular matrix</u> $T = (a_{n,k})$, $\left\{ a_{n,k} \right\}_{k=0}^{n}$ and $\left\{ a_{n-1,k} - a_{n,k+1} \right\}_{k=0}^{n-1}$ <u>are non-negative non-decreasing sequences with respect to</u> k, and $\left\{ a_{n,k+1} - a_{n,k} \right\}_{k=0}^{n-1}$ <u>is a non-decreasing sequence with respect to</u> k, <u>then</u>

$$\sum_{n=2\tau+1}^{\infty} (a_{n,\tau} - a_{n,0}) \leq K ,$$

where $\tau = [K/2t]$, $c > 0$, $0 < t < \pi$.

__Proof.__ Since, by hypotheses

$$a_{n,\tau} - a_{n,0} = \sum_{v=0}^{\tau-1} (a_{n,\tau+1} - a_{n,v})$$

$$\leq \tau (a_{n,\tau} - a_{n,\tau-1})$$

$$\leq \tau (a_{n-1,\tau-1} - a_{n,\tau-1}),$$

we have

$$\sum_{n=2\tau+1}^{\infty} (a_{n,\tau} - a_{n,0}) \leq \tau \sum_{n=2\tau+1}^{\infty} (a_{n-1,\tau-1} - a_{n,\tau-1})$$

$$= \tau (a_{2\tau,\tau-1} - \lim_{n \to \infty} a_{n,\tau-1})$$

$$= \tau . a_{2\tau, \tau-1}$$

$$\leq K . \tau \frac{1}{2\tau - \tau + 2} \leq K ,$$

by Lemma 3.

__Lemma 9.__ __If, for a regular infinite triangular matrix__ $T = (a_{n,k})$, $A_{n,0} = 1$; $\left\{ a_{n,k} \right\}_{k=0}^{n}$ and $\left\{ a_{n-1,k} - a_{n,k+1} \right\}_{k=0}^{n-1}$

are non-negative and non-decreasing sequences with respect to k, then

$$\sum_{n=2\tau+1}^{\infty} (A_{n,\tau} - A_{n-1,\tau-1} + a_{n,0}) \leq K ,$$

where $\tau = [C/2t]$, $C > 0$, $0 < t < \pi$.

Proof. Since, by hypotheses,

$$A_{n,\tau} - A_{n-1,\tau-1} + a_{n,0}$$

$$= \sum_{v=\tau}^{n} a_{n,v} - \sum_{v=\tau-1}^{n-1} a_{n-1,v} + \sum_{v=0}^{n-1} a_{n-1,v} - \sum_{v=1}^{n} a_{n,v}$$

(since $a_{n,0} = A_{n,0} - A_{n,1}$)

$$= \sum_{v=0}^{\tau-2} a_{n-1,v} - \sum_{v=1}^{\tau-1} a_{n,v}$$

$$= \sum_{v=0}^{\tau-2} (a_{n-1,v} - a_{n,v+1})$$

$$\leq (\tau-1)\,(a_{n-1,\tau-2} - a_{n,\tau-1})$$

$$\leq (\tau-1)\,(a_{n-1,\tau-1} - a_{n,\tau-1}),$$

we have,

$$\sum_{n=2\tau+1}^{\infty} (A_{n,\tau} - A_{n-1,\tau-1} + a_{n,o})$$

$$\leq (\tau-1) \sum_{n=2\tau+1}^{\infty} (a_{n-1,\tau-1} - a_{n,\tau-1})$$

$$= (\tau-1) (a_{2\tau,\tau-1} - \lim_{n\to\infty} a_{n,\tau-1})$$

$$= (\tau-1)\, a_{2\tau,\tau-1}$$

$$\leq K (\tau-1) \frac{1}{2\tau-\tau+2} \leq K ,$$

by hypotheses and Lemma 3.

5.6. <u>Proof of Theorem</u> 5.1.

Let

$$v_n = (n+1)\, a_{n,o} ,$$

$$u_n = \lambda_n v_n A_{n+1}(x) ,$$

and

$$s_n^{*} = \sum_{k=o}^{n} u_k ,$$

where

$$A_{n+1}(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \operatorname{Cos}(n+1)t \, dt.$$

Let t_n^* denote the T-transform of the sequence $\{s_n^*\}$. Then, by definition,

$$t_n^* = \sum_{k=0}^{n} a_{n,k} \, s_k^*$$

$$= \sum_{k=0}^{n} A_{n,k} \, u_k.$$

Hence,

$$t_n^* - t_{n-1}^* = \sum_{k=0}^{n} (A_{n,k} - A_{n-1,k}) \, u_k$$

$$= \sum_{k=0}^{n-1} (A_{n,n-k} - A_{n-1,n-k}) \, u_{n-k} \, ,$$

as $A_{n,0} = A_{n-1,0} = 1$.

Therefore, to prove the theorem, we need to show that, under our hypotheses,

$$\frac{\pi}{2} \sum_{n=1}^{\infty} |t_n^* - t_{n-1}^*| = \sum_{n=1}^{\infty} \left| \int_0^\pi \phi(t) \left\{ \sum_{k=0}^{n-1} (A_{n,n-k} - A_{n-1,n-k}) \lambda_{n-k} v_{n-k} \times \operatorname{Cos}(n-k+1)t \right\} dt \right|$$

$$= \sum_{n=1}^{\infty} \left| \int_0^{\pi} \phi(t)\, M(n,t)\, dt \right| < \infty ,$$

where

$$M(n,t) = \sum_{k=0}^{n-1} (A_{n,n-k} - A_{n-1,n-k})\lambda_{n-k} v_{n-k} \mathrm{Cos}(n-k+1)t.$$

Integrating by parts, we have

$$\int_0^{\pi} \phi(t) M(n,t) dt = \left[\phi(t) \int_0^t M(n,u) du\right]_0^{\pi} - \int_0^{\pi} \left\{\int_0^t M(n,u) du\right\} d\phi(t)$$

$$= - \int_0^{\pi} \left\{ \int_0^t M(n,u) du \right\} d\, \phi(t).$$

Therefore,

$$\sum_{n=1}^{\infty} \left| \int_0^{\pi} \phi(t) M(n,t) dt \right| = \sum_{n=1}^{\infty} \left| \left\{ \int_0^{\pi} M(n,u) du \right\} d\, \phi(t) \right|$$

$$\leq \sum_{n=1}^{\infty} \left\{ \int_0^{\pi} |d\phi(t)|\, \left| \int_0^t M(n,u) du \right| \right\}.$$

Now, since, under the hypothesis of the theorem,

$$\int_0^{\pi} \lambda(C/t)\ |d\phi(t)| < \infty ,$$

it is sufficient, for our purpose, to show that, uniformly for $0 < t < \pi$,

$$\sum_{n=1}^{\infty} \left| \int_0^t \Psi(n,u)\, du \right| = O(\lambda(C/t)),$$

or what is the same as

$$(5.6.1) \qquad \Sigma = \sum_{n=1}^{\infty} \left| \sum_{k=0}^{n} (A_{n,n-k} - A_{n-1,n-k})\lambda_{n-k} v_{n-k} \frac{\sin(n-k+1)t}{n-k+1} \right|$$

$$= O(\lambda(C/t)).$$

Let us write $\tau = [C/2t]$ and $m = [n/2]$. Thus

$$\Sigma \leq \sum_{n=1}^{2\tau} \left| \sum_{k=0}^{n-1} (A_{n,n-k} - A_{n-1,n-k})\lambda_{n-k} v_{n-k} \frac{\sin(n-k+1)t}{n-k+1} \right|$$

$$+ \sum_{n=2\tau+1}^{\infty} \left| \sum_{k=0}^{m} (A_{n,n-k} - A_{n-1,n-k})\lambda_{n-k} v_{n-k} \frac{\sin(n-k+1)t}{n-k+1} \right|$$

$$+ \sum_{n=2\tau+1}^{\infty} \left| \sum_{k=m+1}^{n-1} (A_{n,n-k} - A_{n-1,n-k})\lambda_{n-k} v_{n-k} \frac{\sin(n-k+1)t}{n-k+1} \right|$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.}$$

Therefore, in order to establish (5.6.1), it is enough to show that

$$(5.6.2)r \quad \Sigma_r = O\,(\lambda(C/t)), \qquad (r = 1,2,3).$$

Proof of (5.6.2)1 :

Since

$$1 = A_{n,o} = \sum_{v=o}^{n} a_{n,v} \geq (n+1)\, a_{n,o} = v_n \,,$$

$$|\sin\,(n-k+1)t\,| \leq (n-k+1)t \,,$$

and λ_{n-k} is non-increasing with respect to k, we get

$$\Sigma_1 \leq \sum_{n=1}^{2\tau} \lambda_n \sum_{k=o}^{n-1} |(A_{n,n-k} - A_{n-1,n-k})| \; \frac{(n-k+1)t}{(n-k+1)}$$

$$= t \sum_{n=1}^{2\tau} \lambda_n \sum_{k=o}^{n-1} (A_{n,n-k} - A_{n,n-k+1})$$

$$+ t \sum_{n=1}^{\tau} \lambda_n \sum_{k=o}^{n-1} (A_{n-1,n-k} - A_{n,n-k+1})$$

$$= t \sum_{n=1}^{2\tau} \lambda_n A_{n,1} + O\,(t \sum_{n=1}^{2\tau} \lambda_n \sum_{k=1}^{n} \frac{1}{n})$$

$$= O\,(\lambda_{2\tau})$$

$$= O(\lambda(C/t)),$$

by hypotheses and Lemma 3.

<u>Proof of (5.6.2)2</u> :

Since

$$I = \sum_{k=0}^{m} (A_{n,n-k}-A_{n-1,n-k})\, \lambda_{n-k}\, v_{n-k}\; \frac{\sin(n-k+1)t}{n-k+1}$$

$$= \sum_{k=0}^{m} (A_{n,n-k}-A_{n,n-k+1}-a_{n,o})\lambda_{n-k}a_{n-k,o}\, \sin\,(n-k+1)t$$

$$+ \sum_{k=0}^{m} (A_{n,n-k+1}-A_{n-1,n-k}+a_{n,o})\lambda_{n-k}a_{n-k,o}\, \sin(n-k+1)t$$

$$= \sum_{k=0}^{m} \frac{(a_{n,n-k}-a_{n,o})}{n-k}\,(n-k)\lambda_{n-k}a_{n-k,o}\;\; \sin(n-k+1)t$$

$$+ \sum_{k=0}^{m} \frac{(A_{n,n-k+1}-A_{n-1,n-k}+a_{n,o})}{n-k}\,(n-k)\lambda_{n-k}a_{n-k,o}\sin(n-k+1)t$$

$$= I_1 + I_2, \text{ say,}$$

where, by Abel's transformation, we have

$$I_1 = \sum_{k=0}^{m-1} \Delta \left\{ \frac{a_{n,n-k} - a_{n,o}}{(n-k) a_{n,n-k}} (n-k) \lambda_{n-k} a_{n-k,o} \right\} \sum_{\nu=0}^{k} a_{n,n-\nu} \sin(n-\nu+1)t$$

$$+ \frac{a_{n,n-m} - a_{n,o}}{a_{n,n-m}} \lambda_{n-m}\, a_{n-m,o} \sum_{\nu=0}^{m} a_{n,n-\nu} \sin(n-\nu+1)t,$$

so that, by hypotheses and Lemmas 1and 5,

$$|I_1| \leq K\, A_{n,n-\tau} \left[\sum_{k=0}^{m-1} \frac{a_{n,n-k} - a_{n,o}}{n-k} \frac{1}{a_{n,n-k}} \lambda_{n-k}\, a_{n-k,o} \right.$$

$$+ \sum_{k=0}^{m-1} \frac{a_{n,n-k} - a_{n,o}}{(n-k)} \frac{n-k-1}{a_{n,n-k}} \left| \Delta (\lambda_{n-k} a_{n-k,o}) \right|$$

$$+ \sum_{k=0}^{m-1} \frac{a_{n,n-k} - a_{n,o}}{(n-k)} (n-k-1) \lambda_{n-k-1} a_{n-k-1,o} \left| \Delta \left(\frac{1}{a_{n,n-k}} \right) \right|$$

$$+ \sum_{k=0}^{m-1} \Delta \left(\frac{a_{n,n-k} - a_{n,o}}{n-k} \right) (n-k-1) \lambda_{n-k-1} a_{n-k-1,o} \frac{1}{a_{n,n-k-1}}$$

$$\left. + \frac{a_{n,n-m} - a_{n,o}}{a_{n,n-m}} \lambda_{n-m}\, a_{n-m,o} \right]$$

$$\leq K\, A_{n,n-\tau} \left[\frac{a_{n,n} - a_{n,o}}{n} \frac{1}{a_{n,n-m}} \lambda_{n-m}\, a_{n-m,o} \cdot m \right.$$

$$+ \frac{a_{n,n} - a_{n,0}}{n} \frac{(n-1)}{a_{n,n-m}} \sum_{k=0}^{m-1} \left| \Delta \left(\lambda_{n-k} a_{n-k,0} \right) \right|$$

$$+ \frac{a_{n,n} - a_{n,0}}{n} (n-1) \lambda_{n-m} a_{n-m,0} \sum_{k=0}^{m-1} \left| \Delta \left(\frac{1}{a_{n,n-k}} \right) \right|$$

$$+ (n-1) \lambda_{n-m} \frac{a_{n-m,0}}{a_{n,n-m}} \sum_{k=0}^{m-1} \left| \Delta \left(\frac{a_{n,n-k} - a_{n,0}}{n-k} \right) \right|$$

$$+ \frac{a_{n,n-m} - a_{n,0}}{a_{n,n-m}} \lambda_{n-m} a_{n-m,0} \Bigg]$$

$$\leq K A_{n,n-\tau} \Bigg[\frac{a_{n,n} - a_{n,0}}{n} \frac{1}{a_{n,n-m}} \lambda_{n-m} a_{n-m} \cdot m$$

$$+ \frac{a_{n,n} - a_{n,0}}{n} \frac{(n-1)}{a_{n,n-m}} \lambda_{n-m} a_{n-m,0}$$

$$+ \frac{a_{n,n} - a_{n,0}}{n} (n-1) \lambda_{n-m} a_{n-m,0} \frac{1}{a_{n,n-m}}$$

$$+ \frac{a_{n,n} - a_{n,0}}{n} (n-1) \lambda_{n-m} a_{n-m,0} \frac{1}{a_{n,n-m}}$$

$$+ \frac{a_{n,n-m} - a_{n,0}}{a_{n,n-m}} \lambda_{n-m} a_{n-m,0} \Bigg]$$

$$\leq K\, A_{n,n-\tau}\, \lambda_n\, a_{n,o}\ ;$$

and, again, by Abel's transformation, we have

$$I_2 = \sum_{k=o}^{m} \Delta \left\{ \frac{(A_{n,n-k+1} - A_{n-1,n-k} + a_{n,o})}{(n-k)\, a_{n,n-k}}\, (n-k)\lambda_{n-k}\, a_{n-k,o} \right\} \times$$

$$\times \sum_{v=o}^{k} a_{n,n-v} \sin(n-v+1)t$$

$$+ \frac{(A_{n,n-m+1} - A_{n-1,n-m} + a_{n,o})}{a_{n,n-m}}\, \lambda_{n-m}\, a_{n-m,o} \ \times$$

$$\times \sum_{v=o}^{k} a_{n,n-v} \sin(n-v+1)t\ ,$$

so that, following the same analysis as for the estimate of I_1, by hypotheses and Lemmas 1 and 5, we obtain

$$|I_2| \leq K\, A_{n,n-\tau}\, \frac{A_{n,n+1} - A_{n-1,n} + a_{n,o}}{a_{n,n}}\ \ \lambda_n\, a_{n,o}$$

$$\leq K\, A_{n,n-\tau}\, \lambda_n\, a_{n,o}\ ,$$

by virtue of the fact that $A_{n,n+1} = A_{n-1,n} = o$, $\dfrac{a_{n,o}}{a_{n,n}} \leq 1$, we observe that

$$\Sigma_1 = \sum_{n=\tau+1}^{\infty} |I| \leq \sum_{n=\tau+1}^{\infty} (|I_1| + |I_2|)$$

$$\leq K \sum_{n=\tau+1}^{\infty} A_{n,n-\tau} \lambda_n a_{n,0}$$

$$= O(\lambda_\tau)$$

$$= O(\lambda(C/t)),$$

by hypothesis.

This completes the proof of (5.6.2)2.

<u>Proof of (5.6.2)3</u> :

We write

$$\Sigma_3 = \sum_{n=2\tau+1}^{\infty} \left| \sum_{k=m+1}^{n-1} (A_{n,n-k} - A_{n-1,n-k}) \lambda_{n-k} a_{n-k,0} \sin(n-k+1)t \right|$$

$$\leq \sum_{n=2\tau+1}^{\infty} (|I| + |J| + |L| + |M|)$$

$$= \Sigma_{3,1} + \Sigma_{3,2} + \Sigma_{3,3} + \Sigma_{3,4}, \text{ say.}$$

where

$$I = \sum_{k=m+1}^{n-\tau} (A_{n,n-k} - A_{n,n-k+1} - a_{n,o}) \lambda_{n-k} a_{n-k,o} \sin(n-k+1)t,$$

$$J = \sum_{k=n-\tau+1}^{n-1} (A_{n,n-k} - A_{n,n-k+1} - a_{n,o}) \lambda_{n-k} a_{n-k,o} \sin(n-k+1)t,$$

$$L = \sum_{k=m+1}^{n-\tau} (A_{n,n-k+1} - A_{n-1,n-k} + a_{n,o}) \lambda_{n-k} a_{n-k,o} \sin(n-k+1)t,$$

and

$$M = \sum_{k=n-\tau+1}^{n-1} (A_{n,n-k+1} - A_{n-1,n-k} + a_{n,o}) \lambda_{n-k} a_{n-k,o} \sin(n-k+1)t.$$

Therefore, for the proof of (5.6.2)3, it is enough to show that

$$(5.6.3)s \qquad \Sigma_{3,s} = O(\lambda(C/t)) \qquad (s = 1,2,3,4)$$

We proceed to prove these. First we see that

$$\Sigma_{3,2} = \sum_{n=2\tau+1}^{\infty} |J|$$

$$\leq \sum_{n=2\tau+1}^{\infty} \sum_{k=n-\tau+1}^{n-1} \frac{(a_{n,n-k} - a_{n,o})}{n-k} \lambda_{n-k} v_{n-k} |\sin(n-k+1)t \ |$$

$$\leq K \sum_{n=2\tau+1}^{\infty} \sum_{k=n-\tau+1}^{n-1} \frac{a_{n,n-k}-a_{n,o}}{n-k} \lambda_{n-k}$$

(since $v_{n-k} = O(1)$)

$$\leq K \sum_{n=2\tau+1}^{\infty} \frac{(a_{n,\tau-1}-a_{n,o})}{\tau-1} \lambda_{\tau-1}(\tau-1)$$

(by Lemma 5)

$$\leq K \lambda_{\tau} \sum_{n=2\tau+1}^{\infty} (a_{n,\tau}-a_{n,o})$$

$$\leq K \lambda_{\tau} = O(\lambda(c/t)),$$

by hypotheses and Lemma 8, and similarly,

$$\Sigma_{3,4} = \sum_{n=2\ +1}^{\infty} |N|$$

$$\leq K \sum_{n=2\tau+1}^{\infty} \sum_{k=n-\tau+1}^{n-1} \frac{(A_{n,n-k+1}-A_{n-1,n-k}+a_{n,o})}{n-k} \lambda_{n-k}$$

(since $v_{n-k} = O(1)$)

$$\leq K \lambda_{\tau-1} \sum_{n=2\tau+1}^{\infty} \frac{(A_{n,\tau}-A_{n-1,\tau-1}+a_{n,o})}{(\tau-1)} (\tau-1)$$

$$\leq K\lambda_\tau \sum_{n=2\tau+1}^{\infty} (A_{n,\tau} - A_{n-1,\tau-1} + a_{n,0})$$

$$\leq K\lambda_\tau = O(\lambda(C/t)),$$

by hypotheses and Lemma 9.

Next, since, by Abel's transformation,

$$I = \sum_{k=m+1}^{n-\tau-1} \Delta\left\{(a_{n,n-k} - a_{n,0})\lambda_{n-k} a_{n-k,0}\right\} \sum_{v=0}^{k} \sin(n-v+1)t$$

$$+(a_{n,\tau} - a_{n,0})\lambda_\tau a_{\tau,0} \sum_{v=0}^{n-\tau} \sin(n-v+1)t$$

$$- (a_{n,n-m-1} - a_{n,0})\lambda_{n-m-1} a_{n-m-1} \sum_{v=0}^{m} \sin(n-v+1)t,$$

so that

$$|I| \leq \frac{K}{t}\left[\sum_{k=m+1}^{n-\tau-1} \left|\Delta\left\{(a_{n,n-k} - a_{n,0})\lambda_{n-k} a_{n-k,0}\right\}\right|\right.$$

$$+ (a_{n,\tau} - a_{n,0})\lambda_\tau a_{\tau,0}$$

$$\left.+ (a_{n,n-m-1} - a_{n,0})\lambda_{n-m-1} a_{n-m-1,0}\right]$$

$$\le \frac{K}{t} \lambda_\tau a_{\tau,o}(a_{n,\tau} - a_{n,o}) + \frac{K}{t} (a_{n,n-m-1} - a_{n,o}) \lambda_{n-m-1} a_{n-m-1,o}$$

we obtain

$$\Sigma_{3,1} \le \sum_{n=2\tau+1}^{\infty} |I|$$

$$\le \frac{K}{t} \lambda_\tau a_{\tau,o} \sum_{n=2\tau+1}^{\infty} (a_{n,\tau} - a_{n,o})$$

$$+ \frac{K}{t} \sum_{n=2\tau+1}^{\infty} (a_{n,n-m-1} - a_{n,o}) \lambda_{n-m-1} a_{n-m-1,o}$$

$$\le K \lambda_\tau v_\tau + \frac{K}{t} \sum_{n=2\tau+1}^{\infty} a_{n,n-\tau} \lambda_n a_{n,o}$$

(since, for $n \ge 2\tau+1$, $a_{n-m-1} - a_{n,o} < a_{n,n-\tau}$)

$$\le K \lambda_\tau + K \sum_{n=2\tau+1}^{\infty} A_{n,n-\tau} \lambda_n a_{n,o}$$

(since $v_\tau = O(1)$, and by Lemma 7)

$$\le K \lambda_\tau = O(\lambda(C/t)),$$

by hypotheses.

Again, since, by Abel's transformation,

$$L = \sum_{k=m+1}^{n-\tau-1} \Delta\left[(A_{n,n-k+1}-A_{n-1,n-k}+a_{n,o})\lambda_{n-k}a_{n-k,o}\right] \times$$

$$\times \sum_{v=o}^{k} \sin(n-v+1)t$$

$$+(A_{n,\tau+1}-A_{n-1,\tau}+a_{n,o})\lambda_{\tau}a_{\tau,o}\sum_{v=o}^{n-\tau}\sin(n-v+1)t$$

$$+(A_{n,n-m}-A_{n-1,n-m-1}+a_{n,o})\lambda_{n-m-1}a_{n-m-1,o}\sum_{v=o}^{m}\sin(n-v+1)t,$$

so that

$$|L| \leq \frac{K}{t}\left[\sum_{k=m+1}^{n-1}(a_{n-1,n-k-1}-a_{n,n-k})\lambda_{n-k-1}\,a_{n-k-1}\right.$$

$$+\sum_{k=m+1}^{n-1}(A_{n,n-k+1}-A_{n-1,n-k}+a_{n,o})\,|\Delta\lambda_{n-k}|a_{n-k,o}$$

$$+\sum_{k=m+1}^{n-1}(A_{n,n-k+1}-A_{n-1,n-k}+a_{n,o})\,\lambda_{n-k-1}|\Delta a_{n-k,o}|$$

$$+(A_{n,\tau+1}-A_{n-1,\tau}+a_{n,o})\,\lambda_{\tau}a_{\tau,o}$$

$$\left.+(A_{n,n-m}-A_{n-1,n-m-1}+a_{n,o})\,\lambda_{n-m-1}\,a_{n-m-1,o}\right].$$

$$= L_1 + L_2 + L_3 + L_4 + L_5, \text{ say,}$$

and

$$L_{3,5} = \sum_{n=2\tau+1}^{\infty} |L| \leq \sum_{n=2\tau+1}^{\infty} (L_1 + L_2 + L_3 + L_4 + L_5),$$

where, by hypotheses, get

$$\sum_{n=2\tau+1}^{\infty} L_1 \leq \frac{K}{t} \sum_{n=2\tau+1}^{\infty} (a_{n-1,n-m-2} - a_{n,n-m-1}) \sum_{k=m+1}^{n-\tau-1} \lambda_{n-k-1} a_{n-k-1}$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} \lambda_k \, a_{k,0} \sum_{n=2k}^{\infty} (a_{n-1,n-m-2} - a_{n,n-m-1})$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} \lambda_k \, a_{k,0} \sum_{n=2k}^{\infty} (a_{n-1,n-k-2} - a_{n,n-k-1})$$

(since, for $n \geq 2k$, $n-m \leq n-k$),

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} \lambda_k \, a_{k,0} \sum_{n=2k}^{\infty} (a_{n-1,n-\tau-2} - a_{n,n-\tau-1})$$

$$= \frac{K}{t} \sum_{k=\tau}^{\infty} \lambda_k \, a_{k,0} (a_{2k-1,2k-\tau-2} - \lim_{n \to \infty} a_{n,n-\tau-1})$$

$$= \frac{K}{t} \sum_{k=\tau}^{\infty} a_{2k-1,2k-\tau-2} \, \lambda_k \, a_{k,0}$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} a_{k,k-\tau} \, \lambda_k \, a_{k,0}$$

$$\leq K \sum_{k=\tau}^{\infty} A_{k,k-\tau} \lambda_k \, a_{k,0}$$

(by Lemma 7)

$$\leq K \;\; \lambda_\tau = O(\lambda(C/t)) \; ;$$

$$\sum_{n=2\tau+1}^{\infty} L_2 \leq \frac{K}{t} \sum_{n=2\tau+1}^{\infty} \sum_{k=m+1}^{n-\tau-1} \frac{(A_{n,n-k+1} - A_{n-1,n-k} + a_{n,0})}{n-k} \times$$

$$\times \; (n-k) \, a_{n,k,0} \; |\Delta \lambda_{n-k}|$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} k \, a_{k,0} |\Delta\lambda_k| \sum_{n=2k}^{\infty} \frac{(A_{n,n-m} - A_{n-1,n-m-1} + a_{n,0})}{n-m-1}$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} k \, a_{k,0} \; |\Delta\lambda_k| \sum_{n=2k}^{\infty} \frac{(A_{n,n-\tau} - A_{n-1,n-\tau-1} + a_{n,0})}{n-\tau-1}$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} k \, a_{k,0} |\Delta \lambda_k| \sum_{n=2k}^{\infty} (a_{n-1,n-\tau-2} - a_{n,n-\tau-1})$$

$$= \frac{K}{t} \sum_{k=\tau}^{\infty} k \, a_{k,0} \; |\Delta \lambda_k| \; (a_{2k-1,2k-\tau-2} - \lim_{n \to \infty} a_{n,n-\tau-1})$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} a_{2k-1,2k-\tau-2} \, \lambda_k \, a_{k,0}$$

$$\leq \frac{K}{t} \sum_{k=\tau}^{\infty} a_{k,k-\tau} \lambda_k \, a_{k,0}$$

$$\le K \sum_{k=\tau}^{\infty} A_{k,k-\tau}\lambda_k a_{k,0}$$

(by Lemma 7)

$$\le K \lambda_\tau = O(\lambda(C/t)) \;;$$

$$\sum_{n=2\tau+1}^{\infty} L_3 \le \frac{K}{t} \sum_{n=2\tau+1}^{\infty} \sum_{k=n+1}^{\infty} (A_{n,n-k+1}-A_{n-1,n-k}+a_{n,0})\lambda_{n-k-1} |\Delta a_{n-k,0}|$$

$$\le \frac{K}{t} \sum_{k=\tau}^{\infty} k \lambda_k \Delta a_{k,0} \sum_{n=2k}^{\infty} (a_{n-1,n-\tau-2}-a_{n,n-\tau-1})$$

$$\le \frac{K}{t} \sum_{k=\tau}^{\infty} a_{2k-1,2k-\tau-2} \lambda_k a_{k,0}$$

(as in the preceding estimate)

$$\le \frac{K}{t} \sum_{k=\tau}^{\infty} a_{k,k-\tau} \lambda_k a_{k,0}$$

$$\le K \sum_{k=\tau}^{\infty} A_{k,k-\tau} \lambda_k a_{k,0}$$

(by Lemma 7)

$$\le K \lambda_\tau = O(\lambda(C/t)) \;;$$

$$\sum_{n=2\tau+1}^{\infty} L_4 \le \frac{K}{t} \sum_{n=2\tau+1}^{\infty} (A_{n,\tau+1}-A_{n-1,\tau}+a_{n,0})\lambda_\tau a_{\tau,0}$$

$$= \sum_{t}^{K} \lambda_\tau a_{\tau,o} \sum_{n=2\tau+1}^{\infty} (A_{n,\tau+1} - A_{n-1,\tau} + a_{n,o})$$

$$\leq K \lambda_\tau v_\tau \quad \text{(by Lemma 9)}$$

$$\leq K \lambda_\tau = O(\lambda(C/t)); \text{ and}$$

$$\sum_{n=2\tau+1}^{\infty} L_5 \leq \sum_{t}^{K} \sum_{n=2\tau+1}^{\infty} (A_{n,n-m} - A_{n-1,n-m-1} + a_{n,o}) \lambda_{n-m-1} a_{n-m-1,o}$$

$$\leq \sum_{t}^{K} \sum_{n=2\tau+1}^{\infty} (A_{n,n-\tau+1} - A_{n-1,n-\tau} + a_{n,o}) \lambda_n a_{n,o}$$

$$\leq \sum_{t}^{K} \lambda_\tau a_{\tau,o} \sum_{n=2\tau+1}^{\infty} (A_{n,n-\tau+1} - A_{n-1,n-\tau} + a_{n,o})$$

$$\leq K \lambda_\tau v_\tau \quad \text{(by Lemma 9)}$$

$$\leq K \lambda_\tau = O(\lambda(C/t)).$$

Thus, we have

$$L_{3,3} = O(\lambda(C/t)).$$

This proves (5.6.2)3 and thus the proof of the theorem is completed.

Chapter VI

TAUBERIAN THEOREMS FOR $|J,p_n|_k$-SUMMABILITY

6.1 <u>Definitions and Notations</u> : We suppose throughout that

$$p_n > o, \sum_{n=o}^{\infty} p_n = \infty \quad ,$$

and that the radius of convergence of the power series

$$p(x) = \sum_{n=o}^{\infty} p_n x^n, \; p(o) = p_o \; ,$$

is 1. Given any series Σa_n , with the sequence of partial sums $\{s_n\}$, we shall use the notations :

(6.1.1) $\quad p_s(x) = \Sigma p_n s_n x^n, \; (o \leq x < 1)$

and

(6.1.2) $\quad J(x) = J_s(x) = \dfrac{p_s(x)}{p(x)}$.

If the series on the right of (6.1.1) is convergent in the right open interval $[c,1)$, and if

$$J(x) \in BV(c,1), \; (o < c < 1),$$

then the series Σa_n, or the sequence $\{s_n\}$, is said to be absolutely summable (J,p_n), or simply summable $|J,p_n|$, (see [7], [23]). It is said to be summable $|J,p_n|_k$ (see |9|, |86|), $k \geq 1$, if the series (6.1.1) is convergent for $0 \leq x < 1$, and if for $0 < c < 1$, $J(x) \in BV^k(c,1)$.

It is clear that summability $|J,p_n|_1$ is the same as $|J,p_n|$, and for $k > 1$, the summability $|J,p_n|$ and $|J,p_n|_k$ are independent of each other (cf. [56]).

In the special cases, in which

(i) $$p_n = A_n^{\lambda} = \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)\Gamma(\lambda+1)}, \ \lambda > -1, \ n = 0,1,2,\ldots$$

(ii) $p_n = (1+n)^{-1}$, for $n = 0,1,2,\ldots,$

$|J,p_n|_k$-method ($k \geq 1$), reduces respectively to the methods $|A_\lambda|_k$ and $|L|_k$. $|A_\lambda|_1$ and $|L|_1$ are the same as the absolute summability methods, $|A_\lambda|$ and $|L|$ respectively ($|A_0|$ being the absolute Abel summability method).

Now, we write

$$P_n = p_o + p_1 + \ldots + p_n, \; n = o,1,2,\ldots,$$

$$P_{-1} = p_{-1} = o,$$

and

(6.1.3) $\quad \bar{t}^p_n = \dfrac{1}{P_n} \sum\limits_{v=o}^{n} p_v \, s_v, \; (n \geq o).$

The series $\sum\limits_{n=o}^{\infty} a_n$, or the sequence $\{s_n\}$, is said to be summable $| \bar{N}, p_n |_k$, $k \geq 1$, if $\bar{t}^p_n \in BV^k$.

The method $|\bar{N},p_n|_1$ is the same as $|\bar{N},p_n|$-method.

Again, let us write

(6.1.4) $\quad \tilde{t}^p_n = \dfrac{1}{P_n} \sum\limits_{v=1}^{n} P_{v-1} \, a_v \; (P_n \neq o), \; t_o = o.$

Then, we get

(6.1.5) $\quad \bar{\Delta} \bar{t}^p_n = \bar{t}^p_n - \bar{t}^p_{n-1} = \dfrac{p_n}{P_{n-1}} \tilde{t}^p_n , \quad (n \geq 1),$

and

(6.1.6) $\quad s_n = \bar{t}^p_n + \tilde{t}^p_n \quad (n \geq 1).$

Thus, the summability $|\bar{N},p_n|_k$, $k \geq 1$, of $\{s_n\}$ is the same as

(6.1.7) $$\sum_{n=1}^{\infty} n^{k-1} \left| \frac{p_n}{P_{n-1}} \tilde{t}_n^{p} \right|^k < \infty .$$

The summability $|\bar{N},p_n|_k$ of the sequence $\left\{\frac{P_{n-1}a_n}{p_n}\right\}$ is the same as $\left\{\tilde{t}_n^{\,p}\right\} \in BV^k$.

The series $\Sigma\, a_n$, or a sequence $\{s_n\}$, is said to be summable $|C,o|_k$, if $\{s_n\} \in BV^k$.

We use throughout the notations :

(6.1.8) $$G(w) = \frac{\sum\limits_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v)P_u P_{v-u}}{w\left(\sum\limits_{v=o}^{\infty} P_v\, e^{-v/w}\right)^2} ;$$

(6.1.9) $$H(n,w,k) = \left[\frac{\sum\limits_{v=n}^{\infty} e^{-v/n} \sum\limits_{u=n}^{v} (2u-v)\, P_u\, P_{v-u}}{\left(\sum\limits_{v=o}^{\infty} P_v\, e^{-v/w}\right)^2} \right]^{k-1} ;$$

(6.1.10) $$M(w) = \frac{\sum\limits_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v)P_u\, P_{v-u}}{\sum\limits_{v=o}^{\infty} e^{-v/w} \sum\limits_{u=o}^{v} (v-u+1)\, P_u\, P_{v-u}} ;$$

(6.1.11) $\quad m = | w |$,

'C' will be used as a strictly positive constant ; possibly different at each occurrence.

6.2 <u>Introduction</u> : In an attempt to obtain Tauberian theorems for $|J,p_n|$-summability, AHMAD and RAHIMAN ([9]; see also [86]) proved the following theorem which includes some previously known results (see [42]) for $|A|$-summability and yields a new result for $|A_\lambda|$-summability.

<u>Theorem</u> A. If $\Sigma\, a_n$ is summable $|J,p_n|$ and $\{\tilde{t}_n^{\,p}\} \in BV$, and if $\{p_n\}$ is such that

(i) $\quad \dfrac{n\, p_n}{P_{n-1}} < C$, for $n = 1,2,\ldots,$

and

(ii) $\quad M(w) > C$, for $w \geq 1$,

then $\Sigma\, a_n$ is summable $|\bar{N},p_n|$.

RIZVI [90] extended this theorem for $|J,p_n|_k$-summability and obtained the following.

Theorem B. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\{\tilde{t}_n^{p}\} \in BV^k$ and if $\{p_n\}$ is such that

(i) $\dfrac{n\, p_n}{P_{n-1}} < C$, for $n = 1,2,3,\ldots,$

(ii) $\dfrac{\{M(w)\}^k}{w^{k-1}} > C$, for $w \geq 1$,

(iii) $\left(\dfrac{\sum_{v=0}^{\infty} P_v\, e^{-v/3w}}{\sum_{v=0}^{\infty} P_v\, e^{-v/w}} \right)^{k-1}$ is bounded,†

for $w \geq 1$, then Σa_n is summable $|N,p_n|_k$.

Since Theorem A does not cover the important case of $|L|$, very recently, AHMAD and K.C.VARSHNEY ([10]; see also [101]) improved upon Theorem A and established the following theorems. It is interesting to note that these theorems cover both the cases of $|A_\lambda|$, $|L|$-summability methods.

Theorem C. If Σa_n is summable $|J,p_n|$, and $\{\tilde{t}_n^{p}\} \in BV$, and if $\{p_n\}$ is such that

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† This condition is void for $k = 1$.

(i) $\dfrac{n\,p_n}{P_{n-1}} < C$, for $n = 1,2,\ldots$,

and

(ii) $M(w) > C, \dfrac{n\,p_n}{P_{n-1}}$, for $w \geq 1$,

then Σa_n is summable $|N,p_n|$.

<u>Theorem</u> D. If Σa_n is summable $|J,p_n|$, and $\{\tilde{t}_n^{\,p}\} \in BV$, and $\{p_n\}$ satisfies the same conditions as in Theorem C, with condition (ii) replaced by the condition

(ii)' uniformly in $n \geq r \geq 1$,

$$\frac{p_n}{P_{n-1}} = O\left(\frac{p_r}{P_{r-1}} \right) .$$

Then Σa_n is summable $|N,p_n|$

Our object in this chapter is to extend these theorems for $|J,p_n|_k$-summability analogous to the theorems of RIZVI [90].

6.3 We establish the following theorems :

Theorem 6.1. *If, for* $k \geq 1$, Σa_n *is summable* $|J,p_n|_k$, $\{\tilde{t}_n^{\,p}\} \in BV^k$ *and if* $\{p_n\}$ *is such that*

(i) $$\frac{n\,p_n}{P_{n-1}} < C \text{ , for } n = 1,2,\ldots \text{ ,}$$

(ii) $$\frac{\{X(w)\}^k}{w^{k-1}} > C \left\{\frac{m\,p_m}{P_{m-1}}\right\}^k \text{ , for } w \geq 1,$$

and

(iii) $$\left(\frac{\sum_{\nu=0}^{\infty} P_\nu\, e^{-\nu/3w}}{\sum_{\nu=0}^{\infty} P_\nu\, e^{-\nu/w}}\right)^{k-1} \text{ is bounded for } w \geq 1, \text{ §}$$

Then Σa_n *is summable* $|\bar{N},p_n|_k$.

Theorem 6.2. *If, for* $k \geq 1$, Σa_n *is summable* $|J,p_n|_k$, $\{\tilde{t}_n^{\,p}\} \in BV^k$, *and* $\{p_n\}$ *satisfies the same conditions as in Theorem* 6.1, *with condition* (ii) *replaced by the condition*

(ii)' *uniformly in* $n \geq r \geq 1$,

$$\frac{p_n}{P_{n-1}} = O\left(\frac{p_r}{P_{r-1}}\right),$$

§ This condition is void for $k = 1$.

then Σa_n is summable $|\bar{N},p_n|_k$.

We also deduce the following theorems from the above Theorems 6.1 and 6.2 which are analogous to the corresponding results of AHMAD and K.C.VARSHNEY [10].

Theorem 6.3. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\left\{\tilde{t}_n^{\,p}\right\} \in BV^k$, and if $p_n > 0$ and non-increasing sequence, then Σa_n is summable $|\bar{N},p_n|_k$.

Theorem 6.4. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\left\{\tilde{t}_n^{\,p}\right\} \in BV^k$, and $\{p_n\}$ satisfies the same conditions as in Theorem 6.1, then Σa_n is summable $|C,o|_k$.

Theorem 6.5. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\left\{a_n \frac{P_{n-1}}{P_n}\right\} \in BV^k$, and if $\{p_n\}$ satisfies the same conditions as in Theorem 6.1, and furthermore

(iv) $\left(\frac{1}{P_n} \sum_{v=1}^{n} \frac{p_v}{v} \right)^{k-1}$ is bounded for $k \geq 1$, then Σa_n is summable $|C,o|_k$.

6.4 We need the following lemmas for the proof of our theorems.

Lemma 6.1 ([90], *Lemma* 7). *Let*

$$\mu_m > 0,\ \lambda_m = \sum_{n=1}^{m} \mu_n \,,$$

$$d_m = \frac{1}{\lambda_m} \sum_{n=1}^{m} \mu_n c_n \,,$$

and μ_n *is such that, for* $k \geq 1$,

(i) $\left(\dfrac{n\,\mu_n}{\lambda_{n-1}}\right)^{k-1}$ *is bounded* .

(ii) $\left(\dfrac{1}{\lambda_n}\right) \displaystyle\sum_{\nu=1}^{n} \left(\dfrac{\lambda_n}{\nu}\right)^{k-1}$ *is bounded, then*

$\{c_m\} \in BV^k$ *implies* $\{d_m\} \in BV^k$.

Lemma 6.2 ([90], *Lemma* 1). *If* $\{\tilde{t}_n^{\,p}\} \in BV^k$ *and*

$$J(s) = \frac{\sum_{n=0}^{\infty} p_n s_n e^{-ns}}{\sum_{n=0}^{\infty} p_n e^{-ns}} \quad (s > 0),$$

then

$$J'(s) = \frac{\sum_{n=1}^{\infty} \tilde{t}_n^{\,p} \frac{p_n}{P_n} \sum_{\nu=n}^{\infty} e^{-\nu s} \sum_{\mu=0}^{\nu} (2\mu-\nu) p_\mu p_{\nu-\mu}}{\left(\sum_{n=0}^{\infty} p_n e^{-ns}\right)^2} \,.$$

<u>Lemma</u> 6.3. (i) <u>For</u> $v \geq o$, $n \geq o$, ($o \leq v \leq n, n < v \leq 2n$ <u>and</u> $v > 2n$),

$$\sum_{\mu=o}^{n} (2\mu - v-1)\, p_{\mu}\, p_{v-\mu+1} \leq o,$$

(ii) <u>For</u> $n \geq 1$ <u>and</u> $v \geq n$,

$$\sum_{\mu=n}^{v} (2\mu - v)\, p_{\mu}\, p_{v-\mu} \geq o \; .$$

(i) is due to McFADDEN ([57], p.187); (ii) is due to AHMAD and RAHMAN [9], Lemma 3(ii).

<u>Lemma</u> 6.4 ([86], Lemma 5). <u>If</u> $p_n > o$ <u>and</u> $P_n = \sum_{v=o}^{n} p_v$, <u>then</u> $\left\{ P_n / (\sum_{v=o}^{\infty} p_v\, e^{-v/n}) \right\}$ <u>is bounded.</u>

<u>Lemma</u> 6.5. <u>Let</u> $m = [w]$. <u>If</u>

$$\frac{\{H(w)\}^k}{w^{k-1}} > C \left\{ \frac{m\, p_m}{P_{m-1}} \right\}^k$$

<u>for</u> $w \geq 1$, <u>then for</u> $w \geq 1$,

$$\frac{\{G(w)\}^k}{w^{k-1}} > C \left\{ \frac{m\, p_m}{P_{m-1}} \right\}^k, \text{ for } k \geq 1.$$

<u>Proof.</u> The proof is similar to the proof of Lemma 3 of RIZVI [90]. We give it here for the sake of completeness. Since,

$$J_1 < w\,(1- e^{1/w})^{-1} < J_2 \;(\; w \geq 1 \;),$$

where J_1 and J_2 are two suitable positive constants, we observe that,

$$G(w) = \frac{\sum\limits_{n=1}^{\infty} \dfrac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v) p_u\, p_{v-u}}{w(1-e^{-1/w})(\sum\limits_{n=0}^{\infty} e^{-n/w})(\sum\limits_{v=0}^{\infty} e^{-v/w} \sum\limits_{u=0}^{v} p_u p_{v-u}}$$

$$> \frac{1}{w(1-e^{-1/w})} \left\{ \frac{\sum\limits_{n=1}^{\infty} \dfrac{p_n}{P_{n-1}} \sum\limits_{v=0}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v) p_u p_{v-u}}{\sum\limits_{v=0}^{\infty} e^{-v/w} \sum\limits_{u=0}^{v} (v-u+1) p_u\, p_{v-u}} \right\}$$

$$= \frac{1}{w(1-e^{-1/w})}\; M(w)$$

$$> J_1\; M(w),$$

so that, for $k \geq 1$,

$$\frac{\{G(w)\}^k}{w^{k-1}} > J_1^k \frac{\{M(w)\}^k}{w^{k-1}} > C \left\{ \frac{m\, p_m}{P_{m-1}} \right\}^k ,$$

by hypothesis.

This completes the proof of the lemma.

<u>Lemma</u> 6.6([90], Lemma 6). <u>If</u> p_n <u>is such that</u> $\frac{n\, p_n}{P_{n-1}} < C$, <u>then, for</u> $k \geq 1$ <u>and</u> $n \geq 1$,

$$H(n,w,k) = O\left\{(1-e^{-1/w})^{-k+1}\right\}.$$

<u>Lemma</u> 6.7. <u>Let</u> $p_n > o$, $P_n \to \infty$. <u>If</u>

(i) $\frac{n\, p_n}{P_{n-1}} < C$ <u>for</u> $n = 1,2,\ldots,$ <u>where</u> $C > o$,

<u>and</u>

(ii) <u>uniformly in</u> $n \geq r \geq 1$,

$$\frac{p_n}{P_{n-1}} = O\left(\frac{p_r}{P_{r-1}}\right),$$

<u>then, for</u> $w \geq 1$,

$$\frac{\{M(w)\}^k}{w^{k-1}} > C\left(\frac{m\, p_m}{P_{m-1}}\right)^k.$$

The proof is similar to the proof of Lemma 10 in [101].

<u>Lemma</u> 6.8 ([90], Theorem 1). <u>For</u> $\sum_{n=0}^{\infty} a_n$ <u>to be summable</u> $|C,o|_k$ (or $\{s_n\} \in BV^k$), $k \geq 1$, <u>whenever it is summable</u> $|\bar{N},p_n|_k$, $k \geq 1$, <u>it is necessary and sufficient that</u> $\{\tilde{t}_n^p\} \in BV^k$.

6.5 <u>Proof of Theorem</u> 6.1. We use the following alternative definition for $|J,p_n|_k$-summability.

Let

$$J(s) = \frac{\sum_{n=0}^{\infty} p_n\, s_n\, e^{-ns}}{\sum_{n=0}^{\infty} p_n\, e^{-ns}},\ s \geq o.$$

Then the series $\sum_{n=0}^{\infty} a_n$, or the sequence $\{s_n\}$, is said to be summable $|J,p_n|_k$, $k \geq 1$, if the series $\sum_{n=0}^{\infty} p_n\, s_n\, e^{-ns}$ is convergent for $o \leq s < \infty$, and if

$$\int_o^{\infty} (e^s-1)^{k-1}\, |J'(s)|^k\, ds < \infty,\ \text{(See [9], [86]).}$$

We shall suppose throughout, $w = s^{-1}$, $F(w) = \dfrac{1}{w(1-e^{-1/w})}$, $w \geq 1$, and that N is a positive integer.

Since, for two suitable positive constant J_1 and J_2,

$$J_1 < \left\{ w\,(1-e^{-1/w}) \right\}^{-1} < J_2 \quad ,$$

we have

$$\Phi_N = \int_1^N |\widetilde{t}_n^{\,p}|^k \, w^{-1} \left\{ G(w) \right\}^k (e^{1/w}-1)^{k-1} \, dw$$

$$= \sum_{n=1}^{N-1} \int_n^{n+1} |\widetilde{t}_n^{\,p}|^k \, w^{-1} \left\{ G(w) \right\}^k (e^{1/w}-1)^{k-1} \, dw$$

$$= \sum_{n=1}^{N-1} |\widetilde{t}_n^{\,p}|^k \int_n^{n+1} w^{-1} \left\{ G(w) \right\}^k (e^{1/w}-1)^{k-1} \, dw$$

$$\geq \sum_{n=1}^{N-1} |\widetilde{t}_n^{\,p}|^k \int_n^{n+1} \omega^{-1} \left\{ G(w) \right\}^k (1-e^{-1/w})^{k-1} \, dw$$

$$= \sum_{n=1}^{N-1} |\widetilde{t}_n^{\,p}|^k \int_n^{n+1} w^{-1} \, \frac{\left\{ G(w) \right\}^k}{w^{k-1}} \left(\left\{ F(w) \right\}^{-1} \right)^{k-1} dw$$

$$> \frac{C}{J_2^{k-1}} \sum_{n=1}^{N-1} |\widetilde{t}_n^{\,p}|^k \left(\frac{n\, p_n}{P_{n-1}} \right)^k n^{-1} \int_n^{n+1} n\, w^{-1} \, dw$$

(by hypotheses and Lemma 6.5)

$$> C \sum_{n=1}^{N-1} n^{k-1} \, |\widetilde{t}_n^{\,p}|^k \left(\frac{p_n}{P_{n-1}} \right)^k$$

$$\geq C \sum_{m=1}^{N-1} m^{k-1} \mid \overset{*}{\Delta} \tilde{t}_m^{\,p} \mid^{k}$$

by identity (6.1.5). Now, in order to prove the theorem it is sufficient to establish that

$$\phi_N = O(1).$$

We write

$$\phi_N \leq \Bigg[\Big(\int_1^N |\tilde{t}_m^{\,p} \frac{w^{-2/k}}{w^{(k-1)/k}}\, w\, U(w)\, (e^{1/w}-1)^{(k-1)/k}$$

$$- \frac{w^{-2/k}}{w^{(k-1)/k}}\, (e^{1/w}-1)^{(k-1)/k}\, J'(\tfrac{1}{w}) \mid^{k} dw \Big)^{1/k}$$

$$+ \Big(\int_1^N \frac{w^{-2}}{w^{k-1}}\, (e^{1/w}-1)^{k-1} |J'(\tfrac{1}{w})|^{k} dw \Big)^{1/k} \Bigg]^{k}$$

$$= \Big[(S_1)^{1/k} + (S_2)^{1/k} \Big]^{k}, \text{ say.}$$

We will deal with S_1 and S_2 separately. Let us consider first

$$S_2 = \int_1^N \frac{w^{-2}}{w^{k-1}}\, (e^{1/w}-1)^{k-1}\; |J'(\tfrac{1}{w})|^{k}\, dw$$

$$\leq \int_1^N w^{-2}(e^{1/w}-1)^{k-1} \; |J'(\tfrac{1}{w})|^k dw$$

$$= \int_{1/N}^{1} (e^s-1)^{k-1} \; |J'(s)|^k ds = O(1),$$

since Σa_n is summable $|J,p_n|_k$.

Next, since

$$J'(\tfrac{1}{w}) = \frac{\sum\limits_{n=1}^{\infty} \tilde{t}_n^p \frac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v)P_u \, P_{v-u}}{(\sum\limits_{v=0}^{\infty} P_v \, e^{-v/w})^2},$$

we have

$$S_1 = \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{w^{k-1}} \left| \tilde{t}_m^p wG(w) - J'(\tfrac{1}{w})\right|^k dw$$

$$= \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{w^{k-1}} \left| \frac{\sum\limits_{n=1}^{\infty} \frac{p_n}{P_{n-1}} (\tilde{t}_m^p - \tilde{t}_n^p) \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v)P_u P_{v-u}}{(\sum\limits_{v=0}^{\infty} P_v \, e^{-v/w})^2} \right|^k dw$$

$$= \left[\left\{ \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{w^{k-1}} \left(\frac{\sum\limits_{n=1}^{m-1} \frac{p_n}{P_{n-1}} \sum\limits_{r=n+1}^{m} \left| t_r^p \right| \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v)P_u P_{v-u}}{(\sum\limits_{v=0}^{\infty} P_v \, e^{-v/w})^2} \right)^{k} dw \right.\right.$$

$$+ \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{w^{k-1}} \left(\frac{\sum\limits_{n=m+1}^{\infty} \frac{p_n}{P_{n-1}} \sum\limits_{r=m+1}^{n} \left|\Delta \tilde{t}_r^{\,p}\right| \sum\limits_{v=n}^{\infty} e^{-v/wv} \sum\limits_{u=n}^{v} (2u-v) p_u p_{v-u}}{\left(\sum\limits_{v=0}^{\infty} p_v \, e^{-v/w}\right)^2} \right)^k dw \Bigg\}^{\frac{1}{k}} \Bigg]^k$$

$$= \left[(S_{1,1})^{1/k} + (S_{1,2})^{1/k} \right]^k, \text{ say.}$$

$$S_{1,1} = \int_1^N \frac{w^{-2}}{w^{k-1}} (e^{1/w}-1)^{k-1} \left(\frac{\sum\limits_{n=1}^{m-1} \frac{p_n}{P_{n-1}} \sum\limits_{r=n+1}^{m} \left|\Delta \tilde{t}_r^{\,p}\right| \sum\limits_{v=n}^{\infty} e^{-v/wv} \sum\limits_{u=n}^{v} (2u-v) p_u p_{v-u}}{\left(\sum\limits_{v=0}^{\infty} p_v \, e^{-v/w}\right)^2} \right)^k dw$$

$$= \int_1^N \frac{w^{-2}}{w^{k-1}} (e^{1/w}-1)^{k-1} \left(\frac{\sum\limits_{r=2}^{m-1} \left|\Delta \tilde{t}_r^{\,p}\right| \sum\limits_{n=1}^{r-1} \frac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} e^{-v/w} \sum\limits_{u=n}^{v} (2u-v) p_u p_{v-u}}{\left(\sum\limits_{v=0}^{\infty} p_v \, e^{-v/w}\right)^2} \right)^k dw$$

$$\leq \int_1^N \frac{w^{-2}}{w^{k-1}} (e^{1/w}-1)^{k-1} \frac{\sum\limits_{r=2}^{m-1} \left|\Delta \tilde{t}_r^{\,p}\right|^k \left(\sum\limits_{n=1}^{r-1} \frac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} e^{-v/wv} \sum\limits_{u=n}^{v} (2u-v) p_u p_{v-u} \right)^k dw \,\times}{\left(\sum\limits_{v=0}^{n} p_v \, e^{-v/w}\right)^{2k}}$$

$$\times \left(\sum_{r=2}^{m-1} 1 \right)^{k-1}$$

$$\leq \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{\left(\sum\limits_{v=0}^{n} p_v e^{-v/w}\right)^{2k}} \sum_{r=2}^{m-1} \left|\Delta \tilde{t}_r^{\,p}\right|^k \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \left(\sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u-v) p_u p_{v-u} \right)^k dw \,\times$$

$$\times \left(\sum_{n=1}^{r-1} 1 \right)^{k-1}$$

$$\leq K \sum_{r=1}^{N} r^{k-1} \left| \bar{\Delta} \tilde{t}_r^{p} \right| \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \int_r^{\infty} w^{-2}\left(e^{1/w}-1\right)^{k-1} \times$$

$$\times \left(\frac{\sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v\, e^{-v/w}\right)^2} \right)^k dw$$

$$= K \sum_{r=1}^{N} r^{k-1} \left| \bar{\Delta} \tilde{t}_r^{p} \right|^k \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \int_r^{\infty} \frac{1}{w^2} \left(e^{1/w}-1\right)^{k-1} \times$$

$$\times \left(\frac{\sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u-v) P_v P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v\, e^{-v/w}\right)^2} \right) \left(\frac{\sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v\, e^{-v/w}\right)^2} \right)^{k-1} dw$$

$$\leq K \sum_{r=1}^{N} r^{k-1} \left| \bar{\Delta} \tilde{t}_r^{p} \right|^k \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \int_r^{\infty} \frac{e^{\frac{k-1}{w}}\left(1-e^{-1/w}\right)^{k-1}}{w^2\left(1-e^{-1/w}\right)^{k-1}} \times$$

$$\times \left(\frac{\sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u-v) P_v P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v\, e^{-v/w}\right)^2} \right) dw$$

(by Lemma 6.6)

$$\leq K \sum_{r=1}^{N} r^{k-1} \left|\Delta \tilde{t}_r^{\,p}\right|^k \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \int_r^{\infty} \frac{1}{w^2} \left(\frac{\sum_{v=n}^{\infty} e^{-v/w} \sum_{a=n}^{v} (2a-v) p_a P_{v-a}}{\left(\sum_{v=0}^{\infty} P_v \, e^{-v/w}\right)^2} \right) dw$$

$$= -K \sum_{r=1}^{N} r^{k-1} \left|\Delta \tilde{t}_r^{\,p}\right|^k \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \int_r^{\infty} \frac{d}{dw} \left[\frac{\sum_{v=0}^{n-1} P_v \, e^{-v/w}}{\sum_{v=0}^{\infty} P_v \, e^{-v/w}} \right] dw$$

$$= K \sum_{r=1}^{N} r^{k-1} \left|\Delta \tilde{t}_r^{\,p}\right|^k \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \left[- \frac{\sum_{v=0}^{n-1} P_v \, e^{-v/w}}{\sum_{v=0}^{\infty} P_v \, e^{-v/w}} \right]_r^{\infty}$$

$$\leq K \sum_{r=1}^{N} r^{k-1} \left|\Delta \tilde{t}_r^{\,p}\right|^k \sum_{n=1}^{r-1} \left(\frac{p_n}{P_{n-1}}\right)^k \left[\frac{\sum_{v=0}^{n-1} P_v \, e^{-v/r}}{\sum_{v=0}^{\infty} P_v \, e^{-v/r}} \right]$$

$$\leq K \sum_{r=1}^{N} r^{k-1} \left|\Delta \tilde{t}_r^{\,p}\right|^k \sum_{n=1}^{r-1} \frac{1}{n^{k-1}} \, \frac{p_n (1-e^{-n/r})}{\left(\sum_{v=0}^{\infty} P_v e^{-v/r}\right)(1-e^{-1/r})}$$

$$< K \sum_{r=1}^{N} r^{k-1} \left|\Delta \bar{t}_r^{\,p}\right|^k \frac{\sum_{n=0}^{r} p_n}{\sum_{v=0}^{\infty} P_v \, e^{-v/r}}$$

$$\leq K \sum_{\nu=1}^{N} \nu^{k-1} \mid \overset{-}{\Delta} \tilde{t}_\nu{}^p \mid^k \leq K < \infty ,$$

by hypotheses.

Again, since

$$\left(\sum_{r=m+1}^{\infty} e^{(-r/3w \cdot \frac{k}{k-1})} \right)^{k-1} = \left\{ e^{-(m+1)k/3w(k-1)} / (1-e^{-k/3w(k-1)}) \right\}^{k-1}$$

$$\leq \frac{1}{(1-e^{-1/3w})^{k-1}} = \frac{(3w)^{k-1}}{\left\{ 3w(1-e^{-1/3w}) \right\}^{k-1}}$$

$$= \left\{ F(3w) \right\}^{k-1} (3w)^{k-1} = O(w^{k-1}),$$

(since $F(w) \in B$),

and similarly, since

$$\left(\sum_{n=\nu}^{\infty} e^{-n/3w \cdot k/k-1} \right)^{k-1} \text{ is bounded,}$$

we have

$$S_{1,2} \leq \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{w^{k-1}\left(\sum_{\nu=0}^{\infty} P_\nu e^{-\nu/w} \right)^{2k}} \left\{ \sum_{r=m+1}^{\infty} \mid \overset{-}{\Delta} \tilde{t}_r{}^p \mid \sum_{n=r}^{\infty} \left(\frac{p_n}{P_n P_{n-1}} \right) \sum_{\nu=n}^{\infty} e^{-\nu/w} \times \right.$$

$$\left. \times \sum_{u=n}^{\nu} (2u-\nu) P_u P_{\nu-u} \right\}^k dw$$

$$\leq \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{w^{k-1}(\sum\limits_{v=0}^{\infty} P_v e^{-v/w})^{2k}} \Big\{ \sum_{r=m+1}^{\infty} | \Delta \tilde{t}_r^p | e^{-r/3w} \sum_{n=r}^{\infty} \frac{p_n}{P_{n-1}} e^{-n/3w} \times$$

$$\times \sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u} \Big\}^k dw$$

$$\leq \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1} dw}{w^{k-1}(\sum\limits_{v=0}^{\infty} P_v e^{-v/w})^{2k}} \sum_{r=m+1}^{\infty} |\Delta \tilde{t}_r^p|^k \sum_{n=r}^{\infty} (\frac{p_n}{P_{n-1}})^k (\sum_{v=n}^{\infty} e^{-v/3w} \times$$

$$\times \sum_{u=n}^{v} (2u-v) P_u P_{v-u})^k$$

$$\times (\sum_{r=m+1}^{\infty} e^{-r/3w \cdot \frac{k}{k-1}})^{k-1} (\sum_{n=r}^{\infty} e^{-n/3w \cdot k/(k-1)})^{k-1}$$

$$\leq K \int_1^N \frac{w^{-2}(e^{1/w}-1)^{k-1}}{(\sum\limits_{v=0}^{\infty} P_v e^{-v/w})^{2k}} \sum_{r=m+1}^{\infty} r^{k-1} | \Delta \tilde{t}_r^p |^k \sum_{n=r}^{\infty} (\frac{p_n}{P_{n-1}})^k \times$$

$$\times (\sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u})^k dw$$

$$= K \sum_{r=1}^{N} r^{k-1} |\Delta \tilde{t}_r^p|^k \sum_{n=r}^{\infty} (\frac{p_n}{P_{n-1}})^k \int_1^r \frac{(e^{1/w}-1)^{k-1}}{w^2} \times$$

$$\times \Bigg(\frac{\sum\limits_{v=n}^{\infty} e^{-v/3w} \sum\limits_{u=n}^{v} (2u-v) P_u P_{v-u}}{(\sum\limits_{v=0}^{\infty} P_v e^{-v/w})^2} \Bigg)^k dw$$

$$= K \sum_{r=1}^{N} r^{k-1} \left| \bar{\Delta} \tilde{t}_r^{p} \right|^{k} \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right)^{k} \int_1^{r} \frac{(e^{1/w}-1)^{k-1}}{w^2} \times$$

$$\times \left(\frac{\sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v e^{-v/3w}\right)^2} \right)^{k} \left(\frac{\sum_{v=0}^{\infty} P_v e^{-v/3w}}{\left(\sum_{v=0}^{\infty} P_v e^{-v/w}\right)} \right)^{2k} dw$$

$$\leq K \sum_{r=1}^{N} r^{k-1} \left| \bar{\Delta} \tilde{t}_r^{p} \right|^{k} \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right)^{k} \int_1^{r} \frac{(e^{1/w}-1)^{k-1}}{w^2} \times$$

$$\times \left(\frac{\sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v e^{-v/3w}\right)^2} \right)^{k} dw$$

(by hypothesis (iii))

$$= K \sum_{r=1}^{N} r^{k-1} \left| \bar{\Delta} \tilde{t}_r^{p} \right|^{k} \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right)^{k} \int_1^{r} \frac{(e^{1/w}-1)^{k-1}}{w^2} \times$$

$$\times \left(\frac{\sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v e^{-v/3w}\right)^2} \right) \left(\frac{\sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{v} (2u-v) P_v P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v e^{-v/3w}\right)^2} \right)^{k-1} dw$$

$$\leq K \sum_{r=1}^{N} r^{k-1} |\widetilde{\Delta t}_r^{\,p}|^k \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right)^k \int_1^r \frac{e^{(k-1)/w}(1-e^{-1/w})^{k-1}}{w^2(1-e^{-1/3w})^{k-1}} \times$$

$$\times \left(\frac{\sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v e^{-v/3w}\right)^2} \right) dw$$

$$\leq K \sum_{r=1}^{N} r^{k-1} |\widetilde{\Delta t}_r^{\,p}|^k \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right)^k \int_1^r \frac{1}{w^2} \frac{\sum_{v=n}^{\infty} e^{-v/3w} \sum_{u=n}^{\infty} (2u-v) P_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v e^{-v/3w}\right)^2} \, dw$$

$$\left(\text{since, } e^{\frac{k-1}{w}} \left\{ \frac{(1-e^{-1/w})}{(1-e^{-1/3w})} \right\}^{k-1} \in B \right)$$

$$\leq K \sum_{r=1}^{N} r^{k-1} |\widetilde{\Delta t}_r^{\,p}|^k \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right)^k \int_1^r \frac{d}{dw} \left[\frac{\sum_{v=n}^{\infty} P_v e^{-v/3w}}{\sum_{v=0}^{\infty} P_v e^{-v/3w}} \right] dw$$

$$= K \sum_{r=1}^{N} r^{k-1} | \; t_r^{\,p}|^k \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right) \frac{1}{n^{k-1}} \left[\frac{\sum_{v=n}^{\infty} P_v e^{-v/3w}}{\sum_{v=0}^{\infty} P_v e^{-v/3w}} \right]_1^r$$

$$< K \sum_{r=1}^{N} r^{k-1} |\Delta \widetilde{t}_r^{\,p}|^k \sum_{n=r}^{\infty} \left(\frac{p_n}{P_{n-1}}\right) \left[\frac{\sum_{v=n}^{\infty} P_v e^{-v/3r}}{\sum_{v=0}^{\infty} P_v e^{-v/3r}} \right]$$

$$= K \sum_{r=1}^{N} r^{k-1} |\Delta \widetilde{t}_r^{\,p}|^k \frac{\sum_{n=r}^{\infty} \frac{p_n}{P_{n-1}} P_{n-1} e^{-n/3r}}{(1-e^{-1/3r}) \sum_{v=0}^{\infty} P_v e^{-v/3r}} +$$

$$+ K \sum_{r=1}^{N} r^{k-1} |\Delta \widetilde{t}_r^{\,p}|^k \frac{\sum_{n=r}^{\infty} \frac{p_n}{P_{n-1}} \sum_{v=n}^{\infty} p_v e^{-v/3r}}{(\sum_{n=0}^{\infty} P_n e^{-n/3r})(1-e^{-1/3r})}$$

$$\leq K \sum_{r=1}^{N} r^{k-1} |\Delta \widetilde{t}_r^{\,p}|^k \frac{\sum_{n=r}^{\infty} p_n e^{-n/3r}}{\sum_{n=0}^{\infty} p_n e^{-n/3r}} + K \sum_{r=1}^{N} r^{k-1} |\Delta \widetilde{t}_r^{\,p}|^k \times$$

$$\times \frac{\sum_{v=r}^{\infty} v\, p_v e^{-v/3r}}{3r(1-e^{-1/3r}) \sum_{v=0}^{\infty} P_v e^{-v/3r}} =$$

$$= K \sum_{r=1}^{N} r^{k-1} |\Delta \widetilde{t}_r^{\,p}|^k \left(\frac{3r-1}{3r}\right) \frac{\sum_{v=r}^{\infty} p_v e^{-v/3r}}{\sum_{v=0}^{\infty} p_v e^{-v/3r}}$$

$$\leq K \sum_{r=1}^{N} r^{k-1} |\Delta \bar{t}_r^{\,p}|^k + K \sum_{r=1}^{N} r^{k-1} |\Delta \tilde{t}_r^{\,p}|^k \frac{\sum_{v=r}^{\infty} p_v e^{-v/3r}}{3r(1-e^{-1/3r}) \sum_{v=0}^{\infty} p_v e^{-v/3r}}$$

$$\leq K \sum_{r=1}^{N} r^{k-1} \; |\Delta \tilde{t}_r^{\,p}|^k$$

$$\leq K \; ,$$

by hypotheses and by the fact that

$$J_1^* < \frac{1}{3r(1-e^{-1/3r})} < J_2^* \; ,$$

where J_1^* and J_2^* are suitable positive constants.

This completes the proof of Theorem 6.1.

6.6 <u>Proof of Theorem</u> 6.2 <u>through</u> 6.5 : Theorem 6.2 is obtained by combining the results of Theorem 6.1 and Lemma 6.7.

Theorem 6.3 is obtained from Theorem 6.2 by an appeal to Lemma 6.1.

Theorem 6.4 is obtained by combining the result of Theorem 6.1 and Lemma 6.8.

Theorem 6.5 is obtained with the help of Theorem 6.4 by an appeal to Lemma 6.1.

Chapter VII

ON (J,p_n)-SUMMABILITY OF THE DERIVED FOURIER SERIES

7.1 <u>Definitions and Notations</u> : Let $p_n > o$ be such that $\sum_{n=o}^{\infty} p_n = \infty$, and the radius of convergence of the power series,

$$(7.1.1) \qquad p(x) = \sum_{n=o}^{\infty} p_n x^n \; ; \; p(o) = p_o \; ,$$

is 1. Given any series Σa_n, with partial sums s_n, we shall use the notations :

$$(7.1.2) \qquad p_s(x) = \sum_{n=o}^{\infty} p_n s_n x^n$$

and

$$(7.1.3) \qquad J(x) \equiv J_s(x) = p_s(x) \,/\, p(x) .$$

If the series on the right of (7.1.2) is convergent in the open interval (o,1), and if

$$\lim_{x \to 1-o} J_s(x) = s \; ,$$

we say that the series $\sum_{n=0}^{\infty} a_n$, or the sequence $\{s_n\}$ is summable (J,p_n) to the sum s, where s is finite ([13], [40], page 80).

The (J,p_n)-method is regular ([13], [40], p.80).

Particular cases of this method of summability are :

(a) The (A_k)-method : when p_n is given by

$$(1-x)^{-k-1} = \sum_{n=0}^{\infty} p_n x^n, \text{ for } k > -1, \ (|x| < 1),$$

(b) The logarithmic method (L) : when p_n is given by :

$$x^{-1} \log (1-x)^{-1} = \sum_{n=0}^{\infty} p_n x^n, \ (|x| < 1).$$

7.2 Let $f(t)$ be Lebesgue-integrable over $(-\pi,\pi)$ and periodic with period 2π and let the Fourier series of $f(t)$ be

$$(7.2.1) \qquad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$= \sum_{n=0}^{\infty} A_n(t).$$

Then, the first differentiated Fourier series of (7.2.1), at $t = x_0$, is

(7.2.2) $\quad \sum_{n=1}^{\infty} n(b_n \cos nx_0 - a_n \sin nx_0) = \sum_{n=1}^{\infty} n\, B_n(x_0).$

We write

$$\psi(t) = \psi_{x_0}(t) = \left\{f(x_0+t)-f(x_0-t)\right\},$$

$$g(t) = \frac{\psi(t)}{4 \sin \frac{1}{2} t} - C ,$$

$$g_1(t) = \frac{1}{t} \int_0^t g(u)\, du ,$$

where C is a function of x.

7.3 Introduction : Concerning the (L)-summability of series (7.2.2), the following theorem is due to MOHANTY and NANDA [63].

Theorem A. If

$$\int_t^\pi \frac{|g(u)|}{u}\, du = o(\log \tfrac{1}{t}),\ (t \to +0),$$

then the series (7.2.2) is summable (L) to C.

NANDA [66] improved upon Theorem A and proved the following.

Theorem B. If

$$\int_t^\pi \frac{g(u)}{u}\, du = o\ (\log \tfrac{1}{t}),\ (t \to +0),$$

then the series (7.2.2) is summable (L) to C.

NANDA and DAS [67] further improved upon Theorem B in the form of :

Theorem C. If

$$G(t) = \int_t^\pi \frac{g_1(u)}{u}\, du = O(\log \tfrac{1}{t}),\ (t \to +0),$$

then the series (7.2.2) is summable (L) to the sum C.

The object of this chapter is to generalise the above result by proving corresponding theorem for (J,p_n)-summability.

7.4. We prove the following theorem :

Theorem 7.1. *Suppose that*

(i) *the sequence* $\{n\, p_n\}$ *is of bounded variation and*

(ii) *there is an* a, $0 < a < 1$, *such that* $(1-x)^a\, p(x) \downarrow$, as $x \uparrow 1$.

If

$$G(t) = \int_t^\pi \frac{g_1(v)}{v}\, dv = o\,(p(1-t)), \quad (t \to +0),$$

then the series $\Sigma\, n\, B_n(x)$ *is summable* (J,p_n) *to the sum* C.

7.5 We establish the following lemma for the proof of the theorem.

Lemma 7.1. *Let the sequence* $\{p_n\}$ *be such that* $\{n\, p_n\} \in BV$, then

$$\text{(i)} \quad \frac{d}{dt}\left\{t \sum_{n=1}^{\infty} n\, p_n\, x^n \cos(n+\tfrac{1}{2})t\right\} = \begin{cases} O(\frac{1}{1-x}), (t \le (1-x)), \\ O(\frac{1}{t}), (t > (1-x)); \end{cases}$$

$$\text{(ii)} \quad \frac{d^2}{dt^2}\left\{t \sum_{n=1}^{\infty} n\, p_n x^n \cos(n+\tfrac{1}{2})t\right\} = \begin{cases} O(\frac{1}{(1-x)^2}), (t \le (1-x)), \\ O(\frac{1}{t^2}), (t > (1-x)). \end{cases}$$

<u>Proof of the Lemma.</u>

$$\text{(i)} \quad \frac{d}{dt}\left\{t \sum_{n=1}^{\infty} n\, p_n x^n \cos(n+\tfrac{1}{2})t\right\} = \operatorname{Re}\left[\frac{d}{dt}(t \sum_{n=1}^{\infty} np_n x^n\, e^{i(n+\frac{1}{2})t}\right]$$

$$= \operatorname{Re}\left[\frac{d}{dt}(t\, e^{it/2} \sum_{n=1}^{\infty} n\, p_n (x\, e^{it})^n)\right]$$

$$= \operatorname{Re}\left[\frac{d}{dt}\left\{t e^{it/2} (\sum_{n=1}^{m-1} \Delta(np_n) \sum_{\mu=0}^{n} (xe^{it})^{\mu} + \right.\right.$$

$$\left.\left. + \operatorname*{Lt}_{m\to\infty} mp_m \sum_{\mu=0}^{m} (x\, e^{it})^{\mu}\,)\right\}\right]$$

$$= \operatorname{Re}\left[\frac{d}{dt}\left\{t e^{it/2} (\sum_{n=1}^{m-1} \Delta(np_n) (\frac{1-x^{n+1}\, e^{i(n+1)t}}{(1-x\, e^{it}}) + \operatorname*{Lt}_{m\to\infty} mp_m (\frac{1}{1-xe^{it}}))\right\}\right]$$

$$= \mathrm{Re}\left[t\sum_{n=1}^{m-1} \Delta(np_n)\frac{d}{dt}\left(\frac{e^{it/2}(1-x^{n+1}e^{i(n+1)t})}{1-x\,e^{it}}\right) + \right.$$

$$+ \sum_{n=1}^{m-1} \Delta(np_n)\,\frac{e^{it/2}(1-x^{n+1}e^{i(n+1)t})}{1-xe^{it}} +$$

$$\left. + \lim_{m\to\infty} mp_m \frac{e^{it/2}}{1-xe^{it}} + \mathop{\mathrm{Lt}}_{m\to\infty} mp_m\, t\,\frac{d}{dt}\left(\frac{e^{it/2}}{1-xe^{it}}\right)\right]$$

$$= \mathrm{Re}\left[t\sum_{n=1}^{m-1} \Delta(np_n)\left(\frac{(1-xe^{it})\frac{i}{2}e^{it/2}(1-(2n+3)x^{n+1}e^{i(n+1)t})}{(1-x\,e^{it})^2} + \right.\right.$$

$$\left.+\frac{i\,x\,e^{i3t/2}\,(1-x^{n+1}e^{i(n+1)t})}{(1-x\,e^{it})^2}\right)$$

$$+ \sum_{n=1}^{m-1} \Delta(np_n)\,e^{it/2}\,\frac{(1-x^{n+1}e^{i(n+1)t})}{1-xe^{it}} + \mathop{\mathrm{Lt}}_{m\to\infty} mp_m \frac{e^{it/2}}{1-xe^{it}} +$$

$$\left. + \mathop{\mathrm{Lt}}_{m\to\infty} mp_m\,\frac{i}{2}\,te^{it/2}\left(\frac{1+xe^{it}}{(1-xe^{it})^2}\right)\right]$$

Therefore,

$$\left|\frac{d}{dt}\left\{t\sum_{n=1}^{\infty} np_n x^n \cos(n+\tfrac{1}{2})t\right\}\right| \le K\sum_{n=1}^{\infty}|\Delta(np_n)|\frac{t}{|1-xe^{it}|^2}+$$

$$+K\sum_{n=1}^{\infty}|\Delta(np_n)|\frac{1}{|1-xe^{it}|}$$

$$+K\,Lt_{m\to\infty}\, mp_m\frac{t}{|1-xe^{it}|^2}+K\,Lt_{m\to\infty}\, mp_m\frac{1}{|1-xe^{it}|}$$

$$\le K\sum_{n=1}^{\infty}|\Delta(np_n)|\frac{t}{|1-xe^{it}|^2}+K\,Lt_{m\to\infty}\, mp_m\frac{t}{|1-xe^{it}|^2}+$$

$$+K\sum_{n=1}^{\infty}|\Delta(np_n)|\frac{1}{|1-xe^{it}|}+K\,Lt_{m\to\infty}\, mp_m\frac{1}{|1-xe^{it}|}$$

$$=0\left(\frac{t}{|1-xe^{it}|^2}\right)+0\left(\frac{1}{|1-xe^{it}|}\right)$$

$$=0\left(\frac{t}{(1-x)^2+4x\sin^2 t/2}\right)+0\left(\frac{1}{\left\{(1-x)^2+4x\sin^2 t/2\right\}^{1/2}}\right)$$

$$=\left[\begin{array}{ll} 0\left(\dfrac{1}{1-x}\right), & (t\le(1-x)), \\ & \\ 0\left(\dfrac{1}{t}\right), & (t>(1-x))\,; \end{array}\right.$$

(ii) Since, we have

$$\frac{d^2}{dt^2}\left\{t\sum_{n=1}^{\infty} np_n x^n \cos(n+\tfrac{1}{2})t\right\} = \mathrm{Re}\left[\frac{d^2}{dt^2}\left\{t\sum_{n=1}^{\infty} np_n x^n\, e^{i(n+\frac{1}{2})t}\right\}\right]$$

$$=\mathrm{Re}\left[\frac{d^2}{dt^2}\left\{te^{it/2}\sum_{n=1}^{\infty} np_n (xe^{it})^n\right\}\right]$$

$$=\mathrm{Re}\left[\frac{d^2}{dt^2}\left\{te^{it/2}\left(\sum_{n=1}^{m-1}\Delta(np_n)\left(\frac{1-x^{n+1}e^{i(n+1)t}}{1-xe^{it}}\right)+\underset{m\to\infty}{\mathrm{Lt}}\, mp_m\left(\frac{1}{1-xe^{it}}\right)\right)\right\}\right]$$

$$=\mathrm{Re}\left[t\sum_{n=1}^{m-1}\Delta(np_n)\frac{d^2}{dt^2}\left(\frac{e^{it/2}(1-x^{n+1}e^{i(n+1)t})}{1-xe^{it}}\right)+\underset{m\to\infty}{\mathrm{Lt}}\, mp_m\frac{d^2}{dt^2}\left(\frac{te^{it/2}}{1-xe^{it}}\right)+\right.$$

$$\left.+2\sum_{n=1}^{m-1}\Delta(np_n)\frac{d}{dt}\left(\frac{e^{it/2}(1-x^{n+1}e^{i(n+1)t})}{1-x\,e^{it}}\right)\right]$$

$$=\mathrm{Re}\left[t\sum_{n=1}^{m-1}\Delta(np_n)\left\{\frac{-\frac{1}{2}e^{it/2}(1-xe^{it})^2(1-(2n+3)x^n e^{i(n+1)t}(1-(2n+2)e^{\frac{it}{2}})}{(1-xe^{it})^3}\right.\right.$$

$$+\frac{\frac{1}{2}e^{i3t/2}(1-xe^{it})(1-(2n+3)x^{n+1}e^{i(n+1)t}}{(1-xe^{it})^3}+$$

$$+ \frac{(1-xe^{it})^2\left(-\frac{3}{2}xe^{i3t/2}+\frac{x}{2}^{n+2}e^{i(n+5/2)t}(2n+5)\right)}{(1-x\,e^{it})^3} -$$

$$\left. - \frac{2x^2\,e^{i5t/2}(1-x^{n+1}\,e^{i(n+1)t})}{(1-x\,e^{it})^3}\right\} +$$

$$+ \underset{m\to\infty}{Lt}\, mp_m \left\{ \frac{ie^{it/2}(1+xe^{it})}{(1-xe^{it})^2} - \frac{t((4-xe^{it})\frac{x}{2}\,e^{3it/2}+2x^2e^{2it})}{(1-xe^{it})^3} \right\} +$$

$$+2\sum_{n=1}^{m-1}\Delta(np_n)\left\{ \frac{(1-xe^{it})\frac{i}{2}\,e^{it/2}(1-(2n+3)x^{n+1}\,e^{i(n+1)t})}{(1-x\,e^{it})^2} + \right.$$

$$\left.\left. + \frac{i\,x\,e^{i3t/2}(1-x^{n+1}\,e^{i(n+1)t})}{(1-x\,e^{it})^2}\right\}\right]$$

Therefore

$$\left|\frac{d^2}{dt^2}\left[t\sum_{n=1}^{\infty}np_nx^n\cos(n+\tfrac{1}{2})t\right]\right| \le K\sum_{n=1}^{m-1}|\Delta(np_n)|\frac{t}{|1-xe^{it}|^3} + K\underset{m\to\infty}{Lt}\,mp_m\left\{\frac{1}{|1-xe^{it}|^2}+\right\}+$$

$$+K \underset{m\to\infty}{Lt}\ mp_m \frac{t}{|1-xe^{it}|^3} + K\sum_{n=1}^{m-1} |\Delta(np_n)| \frac{1}{|1-xe^{it}|^2}$$

$$\leq K\sum_{n=1}^{m-1} |\Delta(np_n)| \frac{t}{|1-xe^{it}|^3} + K \underset{m\to\infty}{Lt}\ mp_m \frac{t}{|1-xe^{it}|^3} + K\sum_{n=1}^{m-1} |\Delta(np_n)| \frac{1}{|1-xe^{it}|^2}$$

$$+ K \underset{m\to\infty}{Lt}\ mp_m \frac{1}{|1-x\,e^{it}|^2}$$

$$= O\left(\frac{t}{\left\{(1-x)^2+4x\sin^2 t/2\right\}^{3/2}}\right) + O\left(\frac{1}{(1-x)^2+4x\sin^2 t/2}\right)$$

$$= \begin{cases} O\left(\frac{1}{(1-x)^2}\right), & (t < (1-x)), \\ O\left(\frac{1}{t^2}\right), & (t > (1-x)). \end{cases}$$

This completes the proof of the lemma.

7.6 Proof of the Theorem. may assume that $C = 0$. Denoting the n^{th} partial sum of the series (7.2.2) by T_n, we have

$$T_n = \frac{2}{\pi} \int_0^\pi g(t) \frac{\sin nt}{t} dt - \frac{2n}{\pi} \int_0^\pi g(t) \cos (n+\tfrac{1}{2}) t dt + o(1)$$

$$= \frac{2}{\pi} (T_{n,1} - T_{n,2}) + o(1), \text{ say.}$$

Following VARSHNEY [1].

$$J = \sum_{n=o}^{\infty} p_n x^n T_{n,1} = o (p(x)).$$

Now,

$$\sum_{n=o}^{\infty} p_n x^n T_{n,2} = \frac{2}{\pi} \int_0^\pi g(t) \sum_{n=o}^{\infty} np_n x^n \cos (n+ \tfrac{1}{2})t \, dt$$

$$= \frac{2}{\pi} \int_0^\pi g(t) K(t) dt,$$

where

$$K(t) = \sum_{n=1}^{\infty} n p_n x^n \cos (n+ \tfrac{1}{2}) t .$$

Hence, in order to prove the theorem, it is enough to show that

$$I = \int_0^\pi g(t) K(t) dt = o(p(x)).$$

Now, we have

$$G'(t) = -\frac{g_1(t)}{t} .$$

Also, we have

$$g(t) = \frac{d}{dt}(t\ g_1(t)) = g_1(t) + t\ g_1'(t).$$

Now,

$$I = \int_0^\pi \left\{t\ g_1'(t) + g_1(t)\right\} K(t)\ dt$$

$$= \int_0^\pi t\ g_1'(t)\ K(t)dt + \int_0^\pi g_1(t)\ K(t)\ dt$$

$$= I_1 + I_2, \text{ say.}$$

We have

$$I_2 = -\int_0^\pi t\ G'(t)\ K(t)\ dt$$

$$= -\Big[t\ G(t)K(t)\Big]_0^\pi + \int_0^\pi G(t)\frac{d}{dt}\left\{tK(t)\right\}\ dt$$

$$= \int_{0}^{x} G(t) \frac{d}{dt} \left\{ t\ K(t) \right\} dt$$

$$= \int_{0}^{1-x} G(t) \frac{d}{dt} \left\{ t\ K(t) \right\} dt + \int_{1-x}^{x} G(t) \frac{d}{dt} \left\{ t\ K(t) \right\} dt$$

$$= I_{2,1} + I_{2,2}, \text{ say.}$$

Now,

$$I_{2,1} = \int_{0}^{1-x} G(t) \frac{d}{dt} \left\{ t\ K(t) \right\} dt$$

$$= O \left(\frac{1}{(1-x)} \int_{0}^{1-x} |G(t)|\ dt \right)$$

(by part (i) of the lemma)

$$= o \left(\frac{1}{1-x} \int_{0}^{1-x} p(1-t) dt \right),$$

$$= o \left(\frac{1}{1-x}\ p(x)\ (1-x) \right)$$

$$= o\ (p(x)),$$

by hypotheses, since $p(x)$ is increasing as $x \to 1$.

Again,

$$I_{2,2} = \int_{1-x}^{\pi} G(t) \frac{d}{dt} \{t\, K(t)\}\, dt$$

$$= O\left(\int_{1-x}^{\pi} \frac{|G(t)|}{t}\, dt \right)$$

(by part (i) of the lemma)

$$= o\left(\int_{1-x}^{\pi} \frac{p(1-t)}{t}\, dt \right)$$

$$= o\left((1-x)^{a}\, p(x) \int_{1-x}^{\pi} \frac{dt}{t^{a+1}} \right)$$

$$= o\left((1-x)^{a}\, p(x) \frac{1}{(1-x)^{a}} \right)$$

$$= o(\, p(x)\,),$$

by hypotheses,

Integrating by parts, we obtain

$$I_1 = \int_{0}^{\pi} g_1'(t)\ t\, K(t)\, dt$$

$$= \left[g_1(t) t\, K(t) \right]_{0}^{\pi} - \int_{0}^{\pi} g_1(t) \frac{d}{dt} \{t\, K(t)\}\, dt$$

$$= - \int_{0}^{\pi} g_1(t) \frac{d}{dt} \{t\, K(t)\}\, dt$$

$$= \int_0^\pi t\, G'(t) \frac{d}{dt} \left\{ t\, K(t) \right\} dt$$

$$= \left[G(t) t \frac{d}{dt}\, t\, K(t) \right]_0^\pi - \int_0^\pi G(t) \frac{d}{dt} \left\{ t \frac{d}{dt} (t\, K(t)) \right\} dt$$

$$= -\int_0^\pi G(t) \frac{d}{dt} \left\{ t\, K(t) \right\} dt - \int_0^\pi G(t) t \frac{d^2}{dt^2} \left\{ tK(t) \right\} dt$$

$$= - I_{1,1} - I_{1,2}, \text{ say.}$$

Now,

$$I_{1,1} = I_2 = o\ (p(x)), \text{ as } x \to 1 - o,$$

$$I_{1,2} = \int_0^\pi G(t) t \frac{d^2}{dt^2} \left\{ t\, K(t) \right\} dt$$

$$= \int_0^{1-x} G(t) t \frac{d^2}{dt^2} \left\{ tK(t) \right\} dt + \int_{1-x}^\pi G(t) t \frac{d^2}{dt^2} \left\{ tK(t) \right\} dt$$

$$= O\left((1-x) \int_0^{1-x} |G(t)| \frac{dt}{(1-x)^2} \right) + O\left(\int_{1-x}^\pi \frac{|G(t)|}{t}\, dt \right)$$

(by part (ii) of the lemma)

$$= o\left(\frac{1}{(1-x)} p(x)(1-x)\right) + o\left((1-x)^a p(x) \int_{1-x}^\pi \frac{dt}{t^{a+1}} \right)$$

$$= o(p(x)) + o\left((1-x)^a \frac{p(x)}{(1-x)^a} \right)$$

$$= o(p(x)) ,$$

by hypotheses.

This completes the proof of the theorem.

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SUMMARY

of

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THESIS

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ON SOME ASPECTS OF SUMMABILITY AND ITS

APPLICATION

(SUMMARY)

1. Let Σa_n be an infinite series with sequence of partial sums, $\{s_n\}$. Let $B = (b_{nk})$ be an infinite triangular matrix with real or complex elements, so that $b_{nk} = 0$, for every $k > n$. Then

$$B_n = \sum_{k=0}^{n} b_{nk} s_n ,$$

(assuming that 'B_n' exists, for every $n = 0,1,2,\ldots$), defines the matrix transform of the sequence $\{s_n\}$, or the series Σa_n, generated by the elements of the matrix B.

The series Σa_n, or the sequence $\{s_n\}$, is said to be absolutely summable by B-method, or simply summable $|B|$, if $\{B_n\} \in BV$, that is to say,

$$\sum_{n=1}^{\infty} | B_n - B_{n-1} | < \infty ,$$

and it is said to be summable $|B|_k$, $(k \geq 1)$, if $\{B_n\} \in BV^k$,

that is to say,

$$\sum_{n=1}^{\infty} n^{k-1} |b_n - b_{n-1}|^k < \infty .$$

The method $|B|_1$ is the same as $|B|$.

In the special case, in which

$$b_{n,k} = \frac{p_{n-k} q_k}{r_n}, \quad k \leq n ,$$

where $\{p_n\}$ and $\{q_n\}$ are sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + \dots + p_n ; \quad Q_n = q_0 + q_1 + \dots + q_n ;$$

$$r_n = p_n q_0 + p_{n-1} q_1 + \dots + p_0 q_n ,$$

B_n reduces to the generalised Nörlund transform, or (N, p_n, q_n)-transform (see BORWEIN [9]) :

$$t_n^{p,q} = \frac{1}{r_n} \sum_{k=0}^{n} p_{n-k} \, q_k \, s_k \qquad (r_n \neq 0)$$

and $|B|$, $|B|_k$ are then the summability methods $|N, p_n, q_n|$ and $|N, p_n, q_n|_k$ respectively.

In particular, (i) $t_n^{p*q} = t_n^p$, the Nörlund transform, or (N,p_n)-transform, defined by :

$$t_n^p = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}\, s_k \quad (P_n \neq 0),$$

$\bar{t}_n^q$

if $q_n = 1$, for all $n = 0,1,2,\ldots$, and (ii) $t_n^{p*q} = \overline{t_n}^q$, the $(\bar{N},q_n)$-transform, defined by :

$$\overline{t_n}^q = \frac{1}{Q_n} \sum_{k=0}^{n} q_k\, s_k \quad (Q_n \neq 0),$$

when $p_n = 1$, for all $n = 0,1,2,\ldots$. Then the corresponding absolute summability methods are denoted by : $|N,p_n|$, $|N,p_n|_k$ and $|\bar{N},q_n|$, $|\bar{N},q_n|_k$, respectively.

Let $p_n > 0$, such that $\sum_{n=0}^{\infty} p_n = \infty$, and the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \;;\; p(0) = p_0,$$

be 1. Then we shall use the notations :

(*) $$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n$$

and
$$J(x) \equiv J_s(x) = \frac{p_s(x)}{p(x)} .$$

If the series on the right of (*) is convergent for $o \leq x < 1$, and if

$$\lim_{x \to 1-0} J(x) = s , \text{ (s, a finite number)},$$

then the series Σa_n, or the sequence $\{s_n\}$ is said to be summable (J, p_n), to s, (See BORWEIN [7]).

The series Σa_n, or the sequence $\{s_n\}$, is said to be summable $|J, p_n|$, if the series (*) is convergent for $o \leq x < 1$, and " $J(x) \in BV(c,1)$ ", that is,

$$\int_c^1 |J'(x)| dx < \infty, \quad o < c < 1, \quad \text{(see AHMAD [1], [2], DAS [11])}$$

and it is said to be summable $|J, p_n|_k$, $k \geq 1$, if the series (*) is convergent for $o \leq x < 1$, and " $J(x) \in BV^k(c,1)$ ", for $o < c < 1$, that is,

$$\int_c^1 x^{k-1} |J'(x)|^k \, dx < \infty, \quad \text{(See AHMAD and RAHIMAN [4])}$$

The method $|J,p_n|_1$ is the same as $|J,p_n|$, and for $k > 1$, $|J,p_n|$ and $|J,p_n|_k$ are mutually independent. (Cf. MAZHAR [17]).

In the special cases in which

(i) $$p_n = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}, \ \alpha > -1, \ n = 0,1,2,\ldots;$$

(ii) $$p_n = (n+1)^{-1}, \ n = 0,1,2,\ldots ,$$

the methods $|J,p_n|$ and $|J,p_n|_k$ are denoted by $|A_\alpha|$, $|A_\alpha|_k$ ($|A_0|$, $|A_0|_k$ are the same as absolute Abel methods $|A|$ and $|A|_k$ respectively) and $|L|$, $|L|_k$ respectively. (See, e.g. BORWEIN([16], [8]), FLETT [12], MOHANTY and PATNAIK [18], and MAZHAR [17]).

2. Let $f(t)$ be a periodic function with period 2π and integrable -L over $(-\pi,\pi)$. Let

(2.1) $$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t)$$

be the Fourier series of $f(t)$. Then the first differentiated

Fourier series of f is given by

$$(2.2) \qquad \sum_{n=1}^{\infty} n\,(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} n\,B_n(t).$$

The Fourier series and its derived series of $f(t)$, at $t = x$, is given by $\sum_{n=0}^{\infty} A_n(x)$ and $\sum_{n=1}^{\infty} n\,B_n(x)$ and will be denoted by $S[f]_x$ and $S'[f]_x$ respectively.

We write

$$\phi(t) = \frac{1}{2}\left\{f(x+t) + f(x-t)\right\},$$

$$\psi(t) = f(x+t) - f(x-t),$$

$$g(t) = \frac{\psi(t)}{4 \sin \frac{1}{2} t} - C,$$

$$g_1(t) = \frac{1}{t} \int_0^t g(u)\,du,$$

where C is a function of x.

3. We use the following notations:

$$\tau_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu ,$$

$$\tilde{t}_n^p = \frac{1}{P_n} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu ;$$

$$r_n = \sum_{k=0}^{n} p_{n-k} q_k , \quad r_n^* = \sum_{k=0}^{n} \Delta p_{n-k} q_k ;$$

$$r_n^\mu = \sum_{k=\mu}^{n} p_{n-k} q_k , \quad r_n^{*\mu} = \sum_{k=\mu}^{n} \Delta p_{n-k} q_k ;$$

$$R(n,\mu) = (r_n r_n^{*\mu} - r_n^\mu r_n^*) ;$$

$$\sigma_n = \sum_{\mu=1}^{n} \frac{\Delta\mu \left\{ R(n,\mu) \right\}}{r_n r_{n-1}} e_{\mu+1} \tilde{t}_\mu^q .$$

$$\Delta e_n = e_n - e_{n+1} ; \quad \Delta^2 e_n = \Delta (\Delta e_n) = \Delta e_n - \Delta e_{n+1} .$$

$$\lambda(n) = \lambda_n \quad \text{and} \quad n = [w].$$

$$H(w) = \frac{\sum_{n=1}^{\infty} \frac{P_n}{P_{n-1}} \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u-v) P_u P_{v-u}}{\sum_{v=0}^{\infty} e^{-v/w} \sum_{u=0}^{v} (v-u+1) P_u P_{v-u}} ,$$

' a_n is δ-<u>quasimonotone</u>' ; if $a_n \to 0$, and $\Delta a_n \geq -\delta_{n+1}$, where $\{\delta_n\}$ is a sequence of positive numbers.

4. The Thesis consists of eight chapters including a 'Chapter Zero' which contains 'notes on some conventions' followed throughout the thesis. In Chapter I, which is introductory, besides giving further definitions and notations, we give a brief résumé of relevant results which have direct interconnection with our investigations.

5. In Chapter II we prove the following theorems which are generalisationsof the corresponding results of AHMAD [3].

<u>Theorem</u> 1. Let $p_0 > 0$, $p_n \geq 0 (n = 1,2,\ldots)$ and let $\{p_n\}$ be non-increasing. If

$$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\mu_n),$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, and if $\{\epsilon_n\}$ is such that

(a) $\epsilon_n \mu_n = O(1)$, whenever $n \Delta \mu_n = O(\mu_n)$.

(b) there exists a sequence of numbers $\{\alpha_n\}$ such that it is δ-quasi-monotone, with :
(w.r. resp. to δ_n)

$$\sum_{n=1}^{\infty} n \mu_n \delta_n < \infty, \ \Sigma \mu_n \alpha_n < \infty,$$

and

(c) $|\Delta \epsilon_n| \leq \alpha_n$, for all n ,

then the series $\Sigma(n+1)^{-1} P_n \epsilon_n a_n$ is summable $|N,p_n|$.

<u>Theorem</u> 2. Let p_n be the same as in Theorem 1. If the sequence $\{\epsilon_n\}$ is such that

(a) $\log n \, \epsilon_n = O(1)$,

(b) there exists a sequence of numbers $\{\alpha_n\}$ such that it is δ-quasi-monotone with :

$$\sum_{n=2}^{\infty} n \log n \, \delta_n < \infty, \sum_{n=2}^{\infty} \log n \, \alpha_n < \infty ,$$

and

(c) $|\Delta \epsilon_n| \leq \alpha_n$, for all n , then the series $\Sigma(n+1)^{-1}\ p_n\ \epsilon_n\ A_n(x)$ is summable $|N,p_n|$ for almost all values of x.

<u>Theorem</u> 3. Let p_n be the same as in Theorem 1. If $F(x)$ is even, $F(x) \in L^2(-\pi,\pi)$,

$$\int_0^t |F(x)|^2\, dx = O(t), \text{ as } t \to +0,$$

and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem 2, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that the series $\Sigma(n+1)^{-1}\ p_n\ \epsilon_n\ A_n$ is summable $|N,p_n|$.

<u>Theorem</u> 4. Let p_n be the same as in Theorem 1. If $F(x)$ is even, $F(x) \in L(-\pi,\pi)$,

$$\int_0^t |F(x)|\, dx = O(t), \text{ as } t \to +0,$$

and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem 2, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that the series $\Sigma(n+1)^{-1}(\log n)^{-1/2}\ p_n\epsilon_n\ A_n$ is summable $|N,p_n|$.

<u>Theorem</u> 5. Let p_n be the same as in Theorem 1. If $f(z) = \Sigma\, c_n z^n$ is a power series of the complex class L, such that

$$\int_0^t |f(e^{i\theta})|\, d\theta = O(|t|), \text{ as } t \to +0,$$

and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem 2, then the series $\Sigma(n+1)^{-1} P_n\, \epsilon_n\, c_n$ is summable $|N, p_n|$.

6. In Chapter III, a couple of theorems on absolute convergence factors have been obtained, of which include, as a special case, two theorems of DAS ([10], Theorems 1+2), while the first one yields a theorem parallel to a result of PEYERIMHOFF [20] (see Corollary 1 which yields this result). The theorems are :

<u>Theorem</u> 6. Let $\{p_n\}$ and $\{q_n\}$ be such that

(i) $\quad q_{n+1} = O(q_n)$; $n \Delta q_n = O(q_n)$.

(ii) $\quad \sum_{n=0}^{\infty} |c_n| < \infty$,

(iii) $\quad r_n^* = \sum_{v=0}^{n} |p_{n-v}\, q_v| = O(|r_n|)$,

(iv) $\sum_{\nu=0}^{n} |r_\nu - r_{\nu-1}| = O(r_n^*)$,

then the necessary and sufficient condition that $\Sigma\, \epsilon_n a_n$ be absolutely convergent whenever $\Sigma\, a_n$ is summable $|N, p_n, q_n|$, is

$$\epsilon_n r_n = O(1), \text{ as } n \to \infty .$$

<u>Theorem</u> 7. In Theorem 6, if the condition (iii) is dropped, then the condition $\epsilon_n r_n^* = O(1)$, as $n \to \infty$, is sufficient for the absolute convergence of $\Sigma\, \epsilon_n a_n$.

<u>Corollary</u> 1. Let $\{q_n\}$ be such that $q_n > 0$, $q_{n+1} = O(q_n)$ and $n\Delta q_n = O(q_n)$. Then the necessary and sufficient condition that $\sum |\epsilon_n a_n| < \infty$, whenever $\sum a_n$ is summable $|\bar{N}, q_n|$, is that $\epsilon_n = O(q_n/Q_n)$

7. Chapter IV has been devoted to obtain the following two theorems for $|N, p_n, q_n|$-summability factors of infinite series, the second of which include, as a special case, certain results of AHMAD ([3], Theorem 1) and KHAN [15].

<u>Theorem</u> 8. Let $\{p_n\}$, $\{q_n\}$ be positive sequences such that $(n+1)\, q_n \leq K\, Q_n$ † and

† Throughout K denotes an absolute constant, not necessarily the same at each occurrence.

$$\text{(i)} \qquad \sum_{n=\mu}^{\infty} \frac{|R(n,\mu)|}{r_n \, r_{n-1}} \leq K.$$

If

$$\sum_{v=1}^{n} \frac{q_v}{Q_{v-1}} \, |\tilde{t}_v^{\,q}| = O(\lambda_n),$$

where $\{\lambda_n\}$ is a positive monotone non-decreasing sequence, and if the sequences $\{\epsilon_n\}$ and $\{\lambda_n\}$ are such that

$$\text{(ii) (a) } \lambda_n \epsilon_n = O(1), \quad \text{(ii)(b) } \Delta\lambda_n = O\Big(\frac{|\Delta q_n|}{q_n} \lambda_n\Big),$$

$$\text{(iii)} \quad \sum_{n=1}^{\infty} Q_n \, \Big|\Delta\Big(\frac{1}{q_n}\Big)\Big| \, |\lambda_n| \, |\Delta \epsilon_{n+1}| < \infty,$$

$$\text{(iv)} \quad \sum_{n=1}^{\infty} \frac{Q_n}{q_n} \lambda_n \, |\Delta^2 \epsilon_n| < \infty,$$

Then the necessary and sufficient condition for $|N,p_n,q_n|$-summability of the series $\Sigma\, \epsilon_n a_n$ is that

$$\sum_{n=1}^{\infty} |u_n| \leq K.$$

<u>Theorem</u> 9. Let $\{p_n\}$ and $\{q_n\}$ be positive sequences such that $(n+1)q_n \leq K\, Q_n$, and

(i) $r_n^* = \sum_{\nu=0}^{n} \Delta_\nu p_{n-\nu} q_\nu = O(q_n)$,

(ii) $\dfrac{r_{n+1}}{r_n} = O(1)$,

(iii) $\sum_{n=\mu}^{\infty} \dfrac{|R(n,\mu)|}{r_n \, r_{n-1}} \leq K$,

(iv) $\sum_{n=\mu}^{\infty} \dfrac{| \, \Delta_\mu \{R(n,\mu)\}|}{r_n \, r_{n-1}} = O\left(\dfrac{q_\mu}{r_{\mu-1}} \right)$

If

$$\sum_{\mu=1}^{n} \frac{q_\mu}{q_{\mu-1}} \, |\tilde{t}_\mu^{\,q}| = O(\lambda_n),$$

where $\{\lambda_n\}$ is a positive non-decreasing sequence, and if the sequences $\{e_n^*\}$ and $\{\lambda_n\}$ are such that

(v) (a) $\lambda_n \, e_n^* = O(1)$, (b) $\Delta \lambda_n = O\left(\dfrac{|\Delta q_n|}{q_n} \lambda_n\right)$,

(vi) $\sum_{n=1}^{\infty} q_n | \Delta\left(\dfrac{1}{q_n}\right) | \, \lambda_n | \Delta \, e_{n+1}^* | < \infty$,

(vii) $\sum_{n=1}^{\infty} \dfrac{Q_n}{n} \lambda_n | \Delta^2 \, e_n^* | < \infty$, then the series $\Sigma \frac{p_n}{Q_n} \epsilon_n^* a_n$ is summable $|N, p_n, q_n|$.

8. Chapter V concerns with problem of absolute matrix summability factors of a Fourier series. We prove the following theorem which generalizes the results of KANNO [14] (which is itself a generalization of a result of SINGH [22]) in the same manner as KISHORE and HOTA [16] generalise an other earlier result of Singh (see [16]).

<u>Theorem</u> 10. <u>Let</u> $T = (a_{n,k})$ <u>be a regular infinite triangular matrix, and let us write</u> $\sum_{v=k}^{n} a_{n,k} = A_{n,k}$, <u>and assume that</u> $A_{n,o} = 1$ <u>for every</u> $n \geq o$, <u>and that</u>

(i) $a_{2n,2n-k} = O(a_{n,n-k})$; $n \Delta a_{n,o} = O(a_{n,o})$,

(ii) $\left\{a_{n,k}\right\}_{k=o}^{n}$ is a positive sequence such that $\left\{(a_{n,k}-a_{n,o})\, a_{k,o}\right\}_{k=o}^{n}$ is a non-decreasing sequence with respect to k,

(iii) $\left\{a_{n-1,k} - a_{n,k+1}\right\}_{k=o}^{n-1}$ is a non-negative and non-decreasing sequence with respect to k,

(iv) $\left\{a_{n,k+1} - a_{n,k}\right\}_{k=0}^{n-1}$ is a non-decreasing sequence with respect to k.

Further, let $\lambda(t)$, $t > 0$, be a positive non-decreasing function such that, (a) $n \Delta\lambda_n = O(\lambda_n)$, (b) $\left\{\lambda_n a_{n,0}\right\}$ is a non-increasing sequence, and

$$\text{(c)} \qquad \sum_{n=k}^{\infty} A_{n,n-k} \lambda_n a_{n,0} = O(\lambda_k),$$

for every k.

If

$$\int_0^{\pi} \lambda(C/t) \ |d\ \phi(t)| < \infty ,$$

for some constant $C > 0$, holds, then the series

$$\sum_{n=1}^{\infty} \lambda_n \left\{(n+1)a_{n,0}\right\} A_{n+1}(t) \quad ,$$

at $t = x$, is summable $|T|$.

Remark. It is to be noted that the conditions (ii) and (b) together imply that ' $\left\{a_{k,0}\right\}$ is a non-increasing sequence' and $\{a_{nk}\}_{k=0}^{n}$ is a positive non-decreasing sequence with respect to k.

9. Chapter VI deals with Tauberian theorems for $|J,p_n|_k$-summability. The following theorems have been proved.

Theorem 11. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $t_n^p \in BV^k$, and if p_n is such that

(i) $$\frac{n\, p_n}{P_{n-1}} < C^{*} \quad \text{for } n = 0,1,2,\ldots,$$

(ii) $$\frac{\{M(w)\}^k}{w^{k-1}} > C\left\{\frac{m\, p_m}{P_{m-1}}\right\}^k, \text{ for } w \geq 1,$$

and

(iii) $$\left(\frac{\sum_{\nu=0}^{\infty} P_\nu\, e^{-\nu/3w}}{\sum_{\nu=0}^{\infty} P_\nu\, e^{-\nu/w}}\right)^{k-1} \text{ is bounded for } w \geq 1,^{\dagger}$$

then Σa_n is summable $|\bar{N},p_n|_k$.

Theorem 12. If for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\{\tilde{t}_n^p\} \in BV^k$, and if $\{p_n\}$ satisfies the same conditions as in

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* C denotes an strictly positive constant.

† This condition is void for k = 1.

Theorem 11, with condition (ii) replaced by the condition

(ii)' uniformly in $n \geq r \geq 1$,

$$\frac{p_n}{P_{n-1}} = O\left(\frac{p_r}{P_{r-1}}\right),$$

then Σa_n is summable $|\bar{N},p_n|_k$.

__Theorem__ 13. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\{\tilde{t}_n{}^p\} \in BV^k$, and if p_n is a positive, non-increasing sequence, then Σa_n is summable $|\bar{N},p_n|_k$.

__Theorem__ 14. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\{\tilde{t}_n{}^p\} \in BV^k$, and if $\{p_n\}$ satisfies the same conditions as in Theorem 11, then Σa_n is summable $|C,o|_k$.

__Theorem__ 15. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\left\{a_n \frac{P_{n-1}}{p_n}\right\} \in BV^k$, and if p_n satisfies the same conditions as in Theorem 11 and furthermore

(iv) $\left(\frac{1}{P_n} \sum_{\nu=1}^{n} \frac{P_\nu}{\nu}\right)^{k-1}$ is bounded for $k \geq 1$, then

Σa_n is summable $|C, C|_k$.

It is to be noted that Theorem 11 is an improvement over a theorem of RIZVI ([21], Theorem 1) and includes as a particular case when k = 1, a result of AHMAD and VARSHNEY ([5], Theorem 1; this is an improvement upon a result of AHMAD and RAHMAN [4]), while Theorems 12 and 13 give simplified conditions which are easy to apply and Theorems 14 and 15 are deduced from Theorem 11.

It is interesting to note that these theorems cover both the cases of $|A|_k$, $|A_\alpha|_k$ $(-1 < \alpha \leq 0)$ and $|E|_k$-summability unlike Rizvi's theorems. It also contain an extension for $|A|_k$-method, of a result of HYSLOP [13].

10. The seventh and the last chapter contains the following theorem on (J, p_n)-summability of derived Fourier series, which is a generalisation of a result of NANDA and DAS ([19], Theorem 2).

Theorem 16. Suppose that : (i) the sequence $\{n\, p_n\}$ is of bounded variation and that

(ii) there is an a, o < a < 1, such that $(1-x)^a p(x)\downarrow$ as $x \uparrow$.

If

$$\int_t^{\pi} \frac{J_1(u)}{u}\, du = o\,(\,p(1-t)),\ (\,t \to +o).$$

then the series $C'[f]_{x_o}$ is summable (J,p_n) to the sum C.

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